Chas. C. Swafford
304 C.E. Berkeley, Cal.

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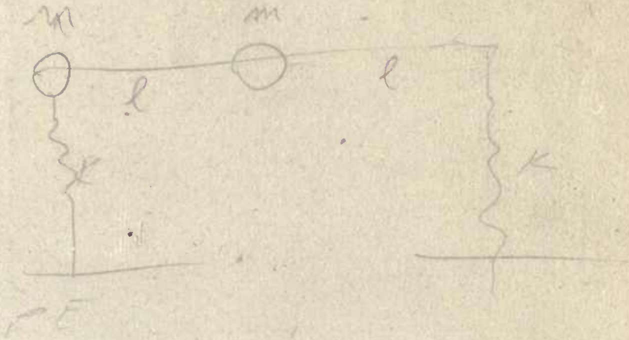

HANDBOOK OF MATHEMATICS FOR ENGINEERS

Handbook of Mathematics for Engineers

BY

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WITH TABLES OF WEIGHTS AND MEASURES BY

LOUIS A. FISCHER, B. S.

CHIEF OF DIVISION OF WEIGHTS AND MEASURES,
U. S. BUREAU OF STANDARDS

REPRINT OF SECTIONS 1 AND 2 OF L. S. MARKS'S
"MECHANICAL ENGINEERS' HANDBOOK"

FIRST EDITION
SECOND IMPRESSION

McGRAW-HILL BOOK COMPANY, INC.
239 WEST 39TH STREET. NEW YORK

LONDON: HILL PUBLISHING CO., LTD.
6 & 8 BOUVERIE ST., E.C.
1918

THE MAPLE PRESS YORK PA

PREFACE

This Handbook of Mathematics is designed to contain, in compact form, accurate statements of those facts and formulas of pure mathematics which are most likely to be useful to the worker in applied mathematics.

It is not intended to take the place of the larger compendiums of pure mathematics on the one hand, or of the technical handbooks of engineering on the other hand; but in its own field it is thought to be more comprehensive than any other similar work in English.

Many topics of an elementary character are presented in a form which permits of immediate utilization even by readers who have had no previous acquaintance with the subject; for example, the practical use of logarithms and logarithmic cross-section paper, and the elementary parts of the modern method of nomography (alignment charts), can be learned from this book without the necessity of consulting separate treatises.

Other sections of the book to which special attention may be called are the chapter on the algebra of complex (or imaginary) quantities, the treatment of the catenary (with special tables), and the brief résumé of the theory of vector analysis.

The mathematical tables (including several which are not ordinarily found) are carried to four significant figures throughout, and no pains have been spared to make them as nearly self-explanatory as possible, even to the reader who makes only occasional use of such tables.

For the Tables of Weights and Measures, which add greatly to its usefulness, the book is indebted to Mr. Louis A. Fischer of the U. S. Bureau of Standards.

All the matter included in the present volume was originally prepared for the Mechanical Engineers' Handbook (Lionel S. Marks, Editor-in-Chief), and was first printed in 1916, as Sections 1 and 2 of that Handbook. The author desires to express his indebtedness to Professor Marks, not only for indispensable advice as to the choice of the topics which would be most useful to engineers, but also for great assistance in many details of the presentation.

All the misprints that have been detected have been corrected in the plates. Notification in regard to any further corrections, and any suggestions toward the improvement or possible enlargement of the book, will be cordially welcomed by the author or the publishers.

E. V. H.

Cambridge, Mass.
April 29, 1918.

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SECTION 1

MATHEMATICAL TABLES
AND
WEIGHTS AND MEASURES

BY

EDWARD V. HUNTINGTON, Ph. D., Associate Professor of Mathematics, Harvard University, Fellow Am. Acad. Arts and Sciences.

LOUIS A. FISCHER, B. S., Chief of Division of Weights and Measures, U. S. Bureau of Standards.

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SQUARES OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.00	1.000	1.002	1.004	1.006	1.008	1.010	1.012	1.014	1.016	1.018	2
1	1.020	1.022	1.024	1.026	1.028	1.030	1.032	1.034	1.036	1.038	
2	1.040	1.042	1.044	1.047	1.049	1.051	1.053	1.055	1.057	1.059	
3	1.061	1.063	1.065	1.067	1.069	1.071	1.073	1.075	1.077	1.080	
4	1.082	1.084	1.086	1.088	1.090	1.092	1.094	1.096	1.098	1.100	
1.05	1.102	1.105	1.107	1.109	1.111	1.113	1.115	1.117	1.119	1.121	
6	1.124	1.126	1.128	1.130	1.132	1.134	1.136	1.138	1.141	1.143	
7	1.145	1.147	1.149	1.151	1.153	1.156	1.158	1.160	1.162	1.164	
8	1.166	1.169	1.171	1.173	1.175	1.177	1.179	1.182	1 184	1.186	
9	1.188	1.190	1.192	1.195	1.197	1.199	1.201	1.203	1.206	1.208	
1.10	1.210	1.212	1.214	1.217	1.219	1.221	1.223	1.225	1.228	1.230	
1	1.232	1.234	1.237	1.239	1.241	1.243	1.245	1.248	1.250	1.252	
2	1.254	1.257	1.259	1.261	1.263	1.266	1.268	1.270	1.272	1.275	
3	1.277	1.279	1.281	1.284	1.286	1.288	1.209	1.293	1.295	1.297	
4	1.300	1.302	1.304	1.306	1.309	1.311	1.313	1.316	1.318	1.320	
1.15	1.322	1.325	1.327	1.329	1.332	1.334	1.336	1.339	1.341	1.343	
6	1.346	1.348	1.350	1 353	1.355	1.357	1.360	1.362	1.364	1.367	
7	1.369	1.371	1.374	1.376	1.378	1.381	1.383	1.385	1.388	1.390	
8	1.392	1.395	1.397	1.399	1.402	1.404	1.407	1.409	1.411	1.414	
9	1.416	1.418	1.421	1.423	1.426	1.428	1.430	1.433	1.435	1.438	
1.20	1.440	1.442	1.445	1.447	1.450	1.452	1.454	1.457	1.459	1.462	
1	1.464	1.467	1.469	1.471	1.474	1.476	1.479	1.481	1.484	1.486	
2	1.488	1.491	1.493	1.496	1.498	1.501	1.503	1.506	1.508	1.510	
3	1.513	1.515	1.518	1.520	1.523	1.525	1.528	1.530	1.533	1.535	
4	1.538	1.540	1.543	1.545	1.548	1.550	1.553	1.555	1.558	1.560	
1.25	1.562	1.565	1.568	1.570	1.573	1.575	1.578	1.580	1.583	1.585	3
6	1.588	1.590	1.593	1.595	1.598	1.600	1.603	1.605	1.608	1.610	
7	1.613	1.615	1.618	1.621	1.623	1.626	1.628	1.631	1.633	1.636	
8	1.638	1.641	1.644	1.646	1.649	1.651	1.654	1.656	1.659	1.662	
9	1.664	1.667	1.669	1.672	1.674	1.677	1.680	1.682	1.685	1.687	
1.30	1.690	1.693	1.695	1.698	1.700	1.703	1.706	1.708	1.711	1.713	
1	1.716	1.719	1.721	1.724	1.727	1.729	1.732	1.734	1.737	1.740	
2	1.742	1.745	1.748	1.750	1.753	1.756	1.758	1.761	1.764	1.766	
3	1.769	1.772	1.774	1.777	1.780	1.782	1.785	1.788	1.790	1.793	
4	1.796	1.798	1.801	1.804	1.806	1.809	1.812	1.814	1.817	1.820	
1.35	1.822	1.825	1.828	1.831	1.833	1.836	1.839	1.841	1.844	1.847	
6	1.850	1.852	1.855	1.858	1.860	1.863	1.866	1.869	1.871	1.874	
7	1.877	1.880	1.882	1.885	1.888	1.891	1.893	1.896	1.899	1.902	
8	1.904	1.907	1.910	1.913	1.915	1.918	1.921	1.924	1.927	1.929	
9	1.932	1.935	1.938	1.940	1.943	1.946	1.949	1.952	1.954	1.957	
1.40	1.960	1.963	1.966	1.968	1.971	1.974	1.977	1.980	1.982	1.985	
1	1.988	1.991	1.994	1.997	1.999	2.002	2.005	2.008	2.011	2.014	
2	2.016	2.019	2.022	2.025	2.028	2.031	2.033	2.036	2.039	2.042	
3	2.045	2.048	2.051	2.053	2.056	2.059	2.062	2.065	2.068	2.071	
4	2.074	2.076	2.079	2.082	2.085	2.088	2.091	2.094	2.097	2.100	
1.45	2.102	2.105	2.108	2.111	2.114	2.117	2.120	2.123	2.126	2.129	
6	2.132	2.135	2.137	2.140	2.143	2.146	2.149	2.152	2.155	2.158	
7	2.161	2.164	2.167	2.170	2.173	2.176	2.179	2.182	2.184	2.187	
8	2.190	2.193	2.196	2.199	2.202	2.205	2.208	2.211	2.214	2.217	
9	2.220	2.223	2.226	2.229	2.232	2.235	2.238	2.241	2.244	2.247	

Moving the decimal point ONE place in *N* requires moving it TWO places in body of table (see p. 6).

SQUARES *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.50	2.250	2.253	2.256	2.259	2.262	2.265	2.268	2.271	2.274	2.277	3
1	2.280	2.283	2.286	2.289	2.292	2.295	2.298	2.301	2.304	2.307	
2	2.310	2.313	2.316	2.320	2.323	2.326	2.329	2.332	2.335	2.338	
3	2.341	2.344	2.347	2.350	2.353	2.356	2.359	2.362	2.365	2.369	
4	2.372	2.375	2.378	2.381	2.384	2.387	2.390	2.393	2.396	2.399	
1.55	2.402	2.406	2.409	2.412	2.415	2.418	2.421	2.424	2.427	2.430	
6	2.434	2.437	2.440	2.443	2.446	2.449	2.452	2.455	2.459	2.462	
7	2.465	2.468	2.471	2.474	2.477	2.481	2.484	2.487	2.490	2.493	
8	2.496	2.500	2.503	2.506	2.509	2.512	2.515	2.519	2.522	2.525	
9	2.528	2.531	2.534	2.538	2.541	2.544	2.547	2.550	2.554	2.557	
1.60	2.560	2.563	2.566	2.570	2.573	2.576	2.579	2.582	2.586	2.589	
1	2.592	2.595	2.599	2.602	2.605	2.608	2.611	2.615	2.618	2.621	
2	2.624	2.628	2.631	2.634	2.637	2.641	2.644	2.647	2.650	2.654	
3	2.657	2.660	2.663	2.667	2.670	2.673	2.676	2.680	2.683	2.686	
4	2.690	2.693	2.696	2.699	2.703	2.706	2.709	2.713	2.716	2.719	
1.65	2.722	2.726	2.729	2.732	2.736	2.739	2.742	2.746	2.749	2.752	
6	2.756	2.759	2.762	2.766	2.769	2.772	2.776	2.779	2.782	2.786	
7	2.789	2.792	2.796	2.799	2.802	2.806	2.809	2.812	2.816	2.819	
8	2.822	2.826	2.829	2.832	2.836	2.839	2.843	2.846	2.849	2.853	
9	2.856	2.859	2.863	2.866	2.870	2.873	2.876	2.880	2.883	2.887	
1.70	2.890	2.893	2.897	2.900	2.904	2.907	2.910	2.914	2.917	2.921	
1	2.924	2.928	2.931	2.934	2.938	2.941	2.945	2.948	2.952	2.955	
2	2.958	2.962	2.965	2.969	2.972	2.976	2.979	2.983	2.986	2.989	
3	2.993	2.996	3.000	3.003	3.007	3.010	3.014	3.017	3.021	3.024	
4	3.028	3.031	3.035	3.038	3.042	3.045	3.049	3.052	3.056	3.059	
1.75	3.062	3.066	3.070	3.073	3.077	3.080	3.084	3.087	3.091	3.094	4
6	3.098	3.101	3.105	3.108	3.112	3.115	3.119	3.122	3.126	3.129	
7	3.133	3.136	3.140	3.144	3.147	3.151	3.154	3.158	3.161	3.165	
8	3.168	3.172	3.176	3.179	3.183	3.186	3.190	3.193	3.197	3.201	
9	3.204	3.208	3.211	3.215	3.218	3.222	3.226	3.229	3.233	3.236	
1.80	3.240	3.244	3.247	3.251	3.254	3.258	3.262	3.265	3.269	3.272	
1	3.276	3.280	3.283	3.287	3.291	3.294	3.298	3.301	3.305	3.309	
2	3.312	3.316	3.320	3.323	3.327	3.331	3.334	3.338	3.342	3.345	
3	3.349	3.353	3.356	3.360	3.364	3.367	3.371	3.375	3.378	3.382	
4	3.386	3.389	3.393	3.397	3.400	3.404	3.408	3.411	3.415	3.419	
1.85	3.422	3.426	3.430	3.434	3.437	3.441	3.445	3.448	3.452	3.456	
6	3.460	3.463	3.467	3.471	3.474	3.478	3.482	3.486	3.489	3.493	
7	3.497	3.501	3.504	3.508	3.512	3.516	3.519	3.523	3.527	3.531	
8	3.534	3.538	3.542	3.546	3.549	3.553	3.557	3.561	3.565	3.568	
9	3.572	3.576	3.580	3.583	3.587	3.591	3.595	3.599	3.602	3.606	
1.90	3.610	3.614	3.618	3.621	3.625	3.629	3.633	3.637	3.640	3.644	
1	3.648	3.652	3.656	3.660	3.663	3.667	3.671	3.675	3.679	3.683	
2	3.686	3.690	3.694	3.698	3.702	3.706	3.709	3.713	3.717	3.721	
3	3.725	3.729	3.733	3.736	3.740	3.744	3.748	3.752	3.756	3.760	
4	3.764	3.767	3.771	3.775	3.779	3.783	3.787	3.791	3.795	3.799	
1.95	3.802	3.806	3.810	3.814	3.818	3.822	3.826	3.830	3.834	3.838	
6	3.842	3.846	3.849	3.853	3.857	3.861	3.865	3.869	3.873	3.877	
7	3.881	3.885	3.889	3.893	3.897	3.901	3.905	3.909	3.912	3.916	
8	3.920	3.924	3.928	3.932	3.936	3.940	3.944	3.948	3.952	3.956	
9	3.960	3.964	3.968	3.972	3.976	3.980	3.984	3.988	3.992	3.996	

$\pi^2 = 9.86960$ $1/\pi^2 = 0.101321$ $e^2 = 7.38906$

SQUARES (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
2.00	4.000	4.004	4.008	4.012	4.016	4.020	4.024	4.028	4.032	4.036	4
1	4.040	4.044	4.048	4.052	4.056	4.060	4.064	4.068	4.072	4.076	
2	4.080	4.084	4.088	4.093	4.097	4.101	4.105	4.109	4.113	4.117	
3	4.121	4.125	4.129	4.133	4.137	4.141	4.145	4.149	4.153	4.158	
4	4.162	4.166	4.170	4.174	4.178	4.182	4.186	4.190	4.194	4.198	
2.05	4.202	4.207	4.211	4.215	4.219	4.223	4.227	4.231	4.235	4.239	
6	4.244	4.248	4.252	4.256	4.260	4.264	4.268	4.272	4.277	4.281	
7	4.285	4.289	4.293	4.297	4.301	4.306	4.310	4.314	4.318	4.322	
8	4.326	4.331	4.335	4.339	4.343	4.347	4.351	4.356	4.360	4.364	
9	4.368	4.372	4.376	4.381	4.385	4.389	4.393	4.397	4.402	4.406	
2.10	4.410	4.414	4.418	4.423	4.427	4.431	4.435	4.439	4.444	4.448	
1	4.452	4.456	4.461	4.465	4.469	4.473	4.477	4.482	4.486	4.490	
2	4.494	4.499	4.503	4.507	4.511	4.516	4.520	4.524	4.528	4.533	
3	4.537	4.541	4.545	4.550	4.554	4.558	4.562	4.567	4.571	4.575	
4	4.580	4.584	4.588	4.592	4.597	4.601	4.605	4.610	4.614	4.618	
2.15	4.622	4.627	4.631	4.635	4.640	4.644	4.648	4.653	4.657	4.661	
6	4.666	4.670	4.674	4.679	4.683	4.687	4.692	4.696	4.700	4.705	
7	4.709	4.713	4.718	4.722	4.726	4.731	4.735	4.739	4.744	4.748	
8	4.752	4.757	4.761	4.765	4.770	4.774	4.779	4.783	4.787	4.792	
9	4.796	4.800	4.805	4.809	4.814	4.818	4.822	4.827	4.831	4.836	
2.20	4.840	4.844	4.849	4.853	4.858	4.862	4.866	4.871	4.875	4.880	
1	4.884	4.889	4.893	4.897	4.902	4.906	4.911	4.915	4.920	4.924	
2	4.928	4.933	4.937	4.942	4.946	4.951	4.955	4.960	4.964	4.968	
3	4.973	4.977	4.982	4.986	4.991	4.995	5.000	5.004	5.009	5.013	
4	5.018	5.022	5.027	5.031	5.036	5.040	5.045	5.049	5.054	5.058	
2.25	5.062	5.067	5.072	5.076	5.081	5.085	5.090	5.094	5.099	5.103	5
6	5.108	5.112	5.117	5.121	5.126	5.130	5.135	5.139	5.144	5.148	
7	5.153	5.157	5.162	5.167	5.171	5.176	5.180	5.185	5.189	5.194	
8	5.198	5.203	5.208	5.212	5.217	5.221	5.226	5.230	5.235	5.240	
9	5.244	5.249	5.253	5.258	5.262	5.267	5.272	5.276	5.281	5.285	
2.30	5.290	5.295	5.299	5.304	5.308	5.313	5.318	5.322	5.327	5.331	
1	5.336	5.341	5.345	5.350	5.355	5.359	5.364	5.368	5.373	5.378	
2	5.382	5.387	5.392	5.396	5.401	5.406	5.410	5.415	5.420	5.424	
3	5.429	5.434	5.438	5.443	5.448	5.452	5.457	5.462	5.466	5.471	
4	5.476	5.480	5.485	5.490	5.494	5.499	5.504	5.508	5.513	5.518	
2.35	5.522	5.527	5.532	5.537	5.541	5.546	5.551	5.555	5.560	5.565	
6	5.570	5.574	5.579	5.584	5.588	5.593	5.598	5.603	5.607	5.612	
7	5.617	5.622	5.626	5.631	5.636	5.641	5.645	5.650	5.655	5.660	
8	5.664	5.669	5.674	5.679	5.683	5.688	5.693	5.698	5.703	5.707	
9	5.712	5.717	5.722	5.726	5.731	5.736	5.741	5.746	5.750	5.755	
2.40	5.760	5.765	5.770	5.774	5.779	5.784	5.789	5.794	5.798	5.803	
1	5.808	5.813	5.818	5.823	5.827	5.832	5.837	5.842	5.847	5.852	
2	5.856	5.861	5.866	5.871	5.876	5.881	5.885	5.890	5.895	5.900	
3	5.905	5.910	5.915	5.919	5.924	5.929	5.934	5.939	5.944	5.949	
4	5.954	5.958	5.963	5.968	5.973	5.978	5.983	5.988	5.993	5.998	
2.45	6.002	6.007	6.012	6.017	6.022	6.027	6.032	6.037	6.042	6.047	
6	6.052	6.057	6.061	6.066	6.071	6.076	6.081	6.086	6.091	6.096	
7	6.101	6.106	6.111	6.116	6.121	6.126	6.131	6.136	6.140	6.145	
8	6.150	6.155	6.160	6.165	6.170	6.175	6.180	6.185	6.190	6.195	
9	6.200	6.205	6.210	6.215	6.220	6.225	6.230	6.235	6.240	6.245	

Moving the decimal point ONE place in *N* requires moving it TWO places in body of table (see p. 6).

SQUARES (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
2.50	6.250	6.255	6.260	6.265	6.270	6.275	6.280	6.285	6.290	6.295	5
1	6.300	6.305	6.310	6.315	6.320	6.325	6.330	6.335	6.340	6.345	
2	6.350	6.355	6.360	6.366	6.371	6.376	6.381	6.386	6.391	6.396	
3	6.401	6.406	6.411	6.416	6.421	6.426	6.431	6.436	6.441	6.447	
4	6.452	6.457	6.462	6.467	6.472	6.477	6.482	6.487	6.492	6.497	
2.55	6.502	6.508	6.513	6.518	6.523	6.528	6.533	6.538	6.543	6.548	
6	6.554	6.559	6.564	6.569	6.574	6.579	6.584	6.589	6.595	6.600	
7	6.605	6.610	6.615	6.620	6.625	6.631	6.636	6.641	6.646	6.651	
8	6.656	6.662	6.667	6.672	6.677	6.682	6.687	6.693	6.698	6.703	
9	6.708	6.713	6.718	6.724	6.729	6.734	6.739	6.744	6.750	6.755	
2.60	6.760	6.765	6.770	6.776	6.781	6.786	6.791	6.796	6.802	6.807	
1	6.812	6.817	6.823	6.828	6.833	6.838	6.843	6.849	6.854	6.859	
2	6.864	6.870	6.875	6.880	6.885	6.891	6.896	6.901	6.906	6.912	
3	6.917	6.922	6.927	6.933	6.938	6.943	6.948	6.954	6.959	6.964	
4	6.970	6.975	6.980	6.985	6.991	6.996	7.001	7.007	7.012	7.017	
2.65	7.022	7.028	7.033	7.038	7.044	7.049	7.054	7.060	7.065	7.070	
6	7.076	7.081	7.086	7.092	7.097	7.102	7.108	7.113	7.118	7.124	
7	7.129	7.134	7.140	7.145	7.150	7.156	7.161	7.166	7.172	7.177	
8	7.182	7.188	7.193	7.198	7.204	7.209	7.215	7.220	7.225	7.231	
9	7.236	7.241	7.247	7.252	7.258	7.263	7.268	7.274	7.279	7.285	
2.70	7.290	7.295	7.301	7.306	7.312	7.317	7.322	7.328	7.333	7.339	
1	7.344	7.350	7.355	7.360	7.366	7.371	7.377	7.382	7.388	7.393	
2	7.398	7.404	7.409	7.415	7.420	7.426	7.431	7.437	7.442	7.447	
3	7.453	7.458	7.464	7.469	7.475	7.480	7.486	7.491	7.497	7.502	
4	7.508	7.513	7.519	7.524	7.530	7.535	7.541	7.546	7.552	7.557	
2.75	7.562	7.568	7.574	7.579	7.585	7.590	7.596	7.601	7.607	7.612	6
6	7.618	7.623	7.629	7.634	7.640	7.645	7.651	7.656	7.662	7.667	
7	7.673	7.678	7.684	7.690	7.695	7.701	7.706	7.712	7.717	7.723	
8	7.728	7.734	7.740	7.745	7.751	7.756	7.762	7.767	7.773	7.779	
9	7.784	7.790	7.795	7.801	7.806	7.812	7.818	7.823	7.829	7.834	
2.80	7.840	7.846	7.851	7.857	7.862	7.868	7.874	7.879	7.885	7.890	
1	7.896	7.902	7.907	7.913	7.919	7.924	7.930	7.935	7.941	7.947	
2	7.952	7.958	7.964	7.969	7.975	7.981	7.986	7.992	7.998	8.003	
3	8.009	8.015	8.020	8.026	8.032	8.037	8.043	8.049	8.054	8.060	
4	8.066	8.071	8.077	8.083	8.088	8.094	8.100	8.105	8.111	8.117	
2.85	8.122	8.128	8.134	8.140	8.145	8.151	8.157	8.162	8.168	8.174	
6	8.180	8.185	8.191	8.197	8.202	8.208	8.214	8.220	8.225	8.231	
7	8.237	8.243	8.248	8.254	8.260	8.266	8.271	8.277	8.283	8.289	
8	8.294	8.300	8.306	8.312	8.317	8.323	8.329	8.335	8.341	8.346	
9	8.352	8.358	8.364	8.369	8.375	8.381	8.387	8.393	8.398	8.404	
2.90	8.410	8.416	8.422	8.427	8.433	8.439	8.445	8.451	8.456	8.462	
1	8.468	8.474	8.480	8.486	8.491	8.497	8.503	8.509	8.515	8.521	
2	8.526	8.532	8.538	8.544	8.550	8.556	8.561	8.567	8.573	8.579	
3	8.585	8.591	8.597	8.602	8.608	8.614	8.620	8.626	8.632	8.638	
4	8.644	8.649	8.655	8.661	8.667	8.673	8.679	8.685	8.691	8.697	
2.95	8.702	8.708	8.714	8.720	8.726	8.732	8.738	8.744	8.750	8.756	
6	8.762	8.768	8.773	8.779	8.785	8.791	8.797	8.803	8.809	8.815	
7	8.821	8.827	8.833	8.839	8.845	8.851	8.857	8.863	8.868	8.874	
8	8.880	8.886	8.892	8.898	8.904	8.910	8.916	8.922	8.928	8.934	
9	8.940	8.946	8.952	8.958	8.964	8.970	8.976	8.982	8.988	8.994	

$\pi^2 = 9.86960$ $1/\pi^2 = 0.101321$ $e^2 = 7.38906$

SQUARES (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
3.00	9.000	9.006	9.012	9.018	9.024	9.030	9.036	9.042	9.048	9.054	6
1	9.060	9.066	9.072	9.078	9.084	9.090	9.096	9.102	9.108	9.114	
2	9.120	9.126	9.132	9.139	9.145	9.151	9.157	9.163	9.169	9.175	
3	9.181	9.187	9.193	9.199	9.205	9.211	9.217	9.223	9.229	9.236	
4	9.242	9.248	9.254	9.260	9.266	9.272	9.278	9.284	9.290	9.296	
3.05	9.302	9.309	9.315	9.321	9.327	9.333	9.339	9.345	9.351	9.357	
6	9.364	9.370	9.376	9.382	9.388	9.394	9.400	9.406	9.413	9.419	
7	9.425	9.431	9.437	9.443	9.449	9.456	9.462	9.468	9.474	9.480	
8	9.486	9.493	9.499	9.505	9.511	9.517	9.523	9.530	9.536	9.542	
9	9.548	9.554	9.560	9.567	9.573	9.579	9.585	9.591	9.598	9.604	
3.10	9.610	9.616	9.622	9.629	9.635	9.641	9.647	9.653	9.660	9.666	
1	9.672	9.678	9.685	9.691	9.697	9.703	9.709	9.716	9.722	9.728	
2	9.734	9.741	9.747	9.753	9.759	9.766	9.772	9.778	9.784	9.791	
3	9.797	9.803	9.809	9.816	9.822	9.828	9.834	9.841	9.847	9.853	
4	9.860	9.866	9.872	9.878	9.885	9.891	9.897	9.904	9.910	9.916	
3.15	9.922	9.929	9.935	9.941	9.948	9.954	9.960	9.967	9.973	9.979	
6	9.986	9.992	9.998	10.005							6
3.1							9.99	10.05	10.11	10.18	6
2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82	
3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49	7
4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18	
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89	
6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	13.62	
7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36	8
8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13	
9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92	
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73	
1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56	
2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40	
3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27	9
4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16	
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07	
6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00	
7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94	10
8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91	
9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90	

$\pi^2 = 9.86960$ $(\pi/2)^2 = 2.46740$ $1/\pi^2 = 0.101321$

Explanation of Table of Squares (pp. 2–7).

This table gives the value of N^2 for values of N from 1 to 10, correct to four figures. (Interpolated values may be in error by 1 in the fourth figure).

To find the square of a number N outside the range from 1 to 10, note that moving the decimal point **one** place in column N is equivalent to moving it **two** places in the body of the table. For example:

$$(3.217)^2 = 10.35; \quad (0.03217)^2 = 0.001035; \quad (3217)^2 = 10350000$$

This table can also be used inversely, to give square roots.

SQUARES *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91	10
1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94	
2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98	
3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05	11
4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14	
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25	
6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38	
7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52	
8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69	12
9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88	
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36.97	37.09	
1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32	
2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56	
3	39.69	39.82	39.94	40.07	40.20	40.32	40.45	40.58	40.70	40.83	13
4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12	
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43	
6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76	
7	44.89	45.02	45.16	45.29	45.43	45.56	45.70	45.83	45.97	46.10	
8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47	14
9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86	
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27	
1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70	
2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14	
3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61	15
4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10	
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61	
6	57.76	57.91	58.06	58.22	58.37	58.52	58.68	58.83	58.98	59.14	
7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68	
8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25	16
9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63.84	
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64.96	65.12	65.29	65.45	
1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08	
2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72	
3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39	17
4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08	
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79	
6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52	
7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26	
8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03	18
9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80.82	
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63	
1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46	
2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30	
3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17	19
4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06	
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97	
6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90	
7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84	
8	96.04	96.24	96.43	96.63	96.83	97.02	97.22	97.42	97.61	97.81	20
9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80	
10.0	100.0										

Moving the decimal point ONE place in *N* requires moving it TWO places in body of table (see p. 6).

CUBES OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.00	1.000	1.003	1.006	1.009	1.012	1.015	1.018	1.021	1.024	1.027	3
1	1.030	1.033	1.036	1.040	1.043	1.046	1.049	1.052	1.055	1.058	
2	1.061	1.064	1.067	1.071	1.074	1.077	1.080	1.083	1.086	1.090	
3	1.093	1.096	1.099	1.102	1.106	1.109	1.112	1.115	1.118	1.122	
4	1.125	1.128	1.131	1.135	1.138	1.141	1.144	1.148	1.151	1.154	
1.05	1.158	1.161	1.164	1.168	1.171	1.174	1.178	1.181	1.184	1.188	
6	1.191	1.194	1.198	1.201	1.205	1.208	1.211	1.215	1.218	1.222	
7	1.225	1.228	1.232	1.235	1.239	1.242	1.246	1.249	1.253	1.256	
8	1.260	1.263	1.267	1.270	1.274	1.277	1.281	1.284	1.288	1.291	4
9	1.295	1.299	1.302	1.306	1.309	1.313	1.317	1.320	1.324	1.327	
1.10	1.331	1.335	1.338	1.342	1.346	1.349	1.353	1.357	1.360	1.364	
1	1.368	1.371	1.375	1.379	1.382	1.386	1.390	1.394	1.397	1.401	
2	1.405	1.409	1.412	1.416	1.420	1.424	1.428	1.431	1.435	1.439	
3	1.443	1.447	1.451	1.454	1.458	1.462	1.466	1.470	1.474	1.478	
4	1.482	1.485	1.489	1.493	1.497	1.501	1.505	1.509	1.513	1.517	
1.15	1.521	1.525	1.529	1.533	1.537	1.541	1.545	1.549	1.553	1.557	
6	1.561	1.565	1.569	1.573	1.577	1.581	1.585	1.589	1.593	1.598	
7	1.602	1.606	1.610	1.614	1.618	1.622	1.626	1.631	1.635	1.639	
8	1.643	1.647	1.651	1.656	1.660	1.664	1.668	1.672	1.677	1.681	
9	1.685	1.689	1.694	1.698	1.702	1.706	1.711	1.715	1.719	1.724	
1.20	1.728	1.732	1.737	1.741	1.745	1.750	1.754	1.758	1.763	1.767	
1	1.772	1.776	1.780	1.785	1.789	1.794	1.798	1.802	1.807	1.811	
2	1.816	1.820	1.825	1.829	1.834	1.838	1.843	1.847	1.852	1.856	
3	1.861	1.865	1.870	1.875	1.879	1.884	1.888	1.893	1.897	1.902	
4	1.907	1.911	1.916	1.920	1.925	1.930	1.934	1.939	1.944	1.948	5
1.25	1.953	1.958	1.963	1.967	1.972	1.977	1.981	1.986	1.991	1.996	
6	2.000	2.005	2.010	2.015	2.019	2.024	2.029	2.034	2.039	2.044	
7	2.048	2.053	2.058	2.063	2.068	2.073	2.078	2.082	2.087	2.092	
8	2.097	2.102	2.107	2.112	2.117	2.122	2.127	2.132	2.137	2.142	
9	2.147	2.152	2.157	2.162	2.167	2.172	2.177	2.182	2.187	2.192	
1.30	2.197	2.202	2.207	2.212	2.217	2.222	2.228	2.233	2.238	2.243	
1	2.248	2.253	2.258	2.264	2.269	2.274	2.279	2.284	2.290	2.295	
2	2.300	2.305	2.310	2.316	2.321	2.326	2.331	2.337	2.342	2.347	
3	2.353	2.358	2.363	2.369	2.374	2.379	2.385	2.390	2.395	2.401	
4	2.406	2.411	2.417	2.422	2.428	2.433	2.439	2.444	2.449	2.455	
1.35	2.460	2.466	2.471	2.477	2.482	2.488	2.493	2.499	2.504	2.510	6
6	2.515	2.521	2.527	2.532	2.538	2.543	2.549	2.554	2.560	2.566	
7	2.571	2.577	2.583	2.588	2.594	2.600	2.605	2.611	2.617	2.622	
8	2.628	2.634	2.640	2.645	2.651	2.657	2.663	2.668	2.674	2.680	
9	2.686	2.691	2.697	2.703	2.709	2.715	2.721	2.726	2.732	2.738	
1.40	2.744	2.750	2.756	2.762	2.768	2.774	2.779	2.785	2.791	2.797	
1	2.803	2.809	2.815	2.821	2.827	2.833	2.839	2.845	2.851	2.857	
2	2.863	2.869	2.875	2.881	2.888	2.894	2.900	2.906	2.912	2.918	
3	2.924	2.930	2.936	2.943	2.949	2.955	2.961	2.967	2.974	2.980	
4	2.986	2.992	2.998	3.005	3.011	3.017	3.023	3.030	3.036	3.042	
1.45	3.049	3.055	3.061	3.068	3.074	3.080	3.087	3.093	3.099	3.106	
6	3.112	3.119	3.125	3.131	3.138	3.144	3.151	3.157	3.164	3.170	
7	3.177	3.183	3.190	3.196	3.203	3.209	3.216	3.222	3.229	3.235	
8	3.242	3.248	3.255	3.262	3.268	3.275	3.281	3.288	3.295	3.301	7
9	3.308	3.315	3.321	3.328	3.335	3.341	3.348	3.355	3.362	3.368	

Moving the decimal point ONE place in *N* requires moving it THREE places in body of table (see p. 10).

CUBES (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.50	3.375	3.382	3.389	3.395	3.402	3.409	3.416	3.422	3.429	3.436	7
1	3.443	3.450	3.457	3.464	3.470	3.477	3.484	3.491	3.498	3.505	
2	3.512	3.519	3.526	3.533	3.540	3.547	3.554	3.561	3.568	3.575	
3	3.582	3.589	3.596	3.603	3.610	3.617	3.624	3.631	3.638	3.645	
4	3.652	3.659	3.667	3.674	3.681	3.688	3.695	3.702	3.709	3.717	
1.55	3.724	3.731	3.738	3.746	3.753	3.760	3.767	3.775	3.782	3.789	
6	3.796	3.804	3.811	3.818	3.826	3.833	3.840	3.848	3.855	3.863	
7	3.870	3.877	3.885	3.892	3.900	3.907	3.914	3.922	3.929	3.937	
8	3.944	3.952	3.959	3.967	3.974	3.982	3.989	3.997	4.005	4.012	8
9	4.020	4.027	4.035	4.042	4.050	4.058	4.065	4.073	4.081	4.088	
1.60	4.096	4.104	4.111	4.119	4.127	4.135	4.142	4.150	4.158	4.166	
1	4.173	4.181	4.189	4.197	4.204	4.212	4.220	4.228	4.236	4.244	
2	4.252	4.259	4.267	4.275	4.283	4.291	4.299	4.307	4.315	4.323	
3	4.331	4.339	4.347	4.355	4.363	4.371	4.379	4.387	4.395	4.403	
4	4.411	4.419	4.427	4.435	4.443	4.451	4.460	4.468	4.476	4.484	
1.65	4.492	4.500	4.508	4.517	4.525	4.533	4.541	4.550	4.558	4.566	
6	4.574	4.583	4.591	4.599	4.607	4.616	4.624	4.632	4.641	4.649	
7	4.657	4.666	4.674	4.683	4.691	4.699	4.708	4.716	4.725	4.733	
8	4.742	4.750	4.759	4.767	4.776	4.784	4.793	4.801	4.810	4.818	
9	4.827	4.835	4.844	4.853	4.861	4.870	4.878	4.887	4.896	4.904	9
1.70	4.913	4.922	4.930	4.939	4.948	4.956	4.965	4.974	4.983	4.991	
1	5.000	5.009	5.018	5.027	5.035	5.044	5.053	5.062	5.071	5.080	
2	5.088	5.097	5.106	5.115	5.124	5.133	5.142	5.151	5.160	5.169	
3	5.178	5.187	5.196	5.205	5.214	5.223	5.232	5.241	5.250	5.259	
4	5.268	5.277	5.286	5.295	5.304	5.314	5.323	5.332	5.341	5.350	
1.75	5.359	5.369	5.378	5.387	5.396	5.405	5.415	5.424	5.433	5.442	
6	5.452	5.461	5.470	5.480	5.489	5.498	5.508	5.517	5.526	5.536	
7	5.545	5.555	5.564	5.573	5.583	5.592	5.602	5.611	5.621	5.630	10
8	5.640	5.649	5.659	5.668	5.678	5.687	5.697	5.707	5,716	5.726	
9	5.735	5.745	5.755	5.764	5.774	5.784	5.793	5.803	5.813	5.822	
1.80	5.832	5.842	5.851	5.861	5.871	5.881	5.891	5.900	5.910	5.920	
1	5.930	5.940	5.949	5.959	5.969	5.979	5.989	5.999	6.009	6.019	
2	6.029	6.039	6.048	6.058	6.068	6.078	6.088	6.098	6.108	6.118	
3	6.128	6.139	6.149	6.159	6.169	6.179	6.189	6.199	6.209	6.219	
4	6.230	6.240	6.250	6.260	6.270	6.280	6.291	6.301	6.311	6.321	
1.85	6.332	6.342	6.352	6.362	6.373	6.383	6.393	6.404	6.414	6.424	
6	6.435	6.445	6.456	6.466	6.476	6.487	6.497	6.508	6.518	6.529	
7	6.539	6.550	6.560	6.571	6.581	6.592	6.602	6.613	6.623	6.634	11
8	6.645	6.655	6.666	6.677	6.687	6.698	6.708	6.719	6.730	6.741	
9	6.751	6.762	6.773	6.783	6.794	6.805	6.816	6.827	6.837	6.848	
1.90	6.859	6.870	6.881	6.892	6.902	6.913	6.924	6.935	6.946	6.957	
1	6.968	6.979	6.990	7.001	7.012	7.023	7.034	7.045	7.056	7.067	
2	7.078	7.089	7.100	7.111	7.122	7.133	7.144	7.156	7.167	7.178	
3	7.189	7.200	7.211	7.223	7.234	7.245	7.256	7.268	7.279	7.290	
4	7.301	7.313	7.324	7.335	7.347	7.358	7.369	7.381	7.392	7.403	
1.95	7.415	7.426	7.438	7.449	7.461	7.472	7.484	7.495	7.507	7.518	12
6	7.530	7.541	7.553	7.564	7.576	7.587	7.599	7.610	7.622	7.634	
7	7.645	7.657	7.669	7.680	7.692	7.704	7.715	7.727	7.739	7.751	
8	7.762	7.774	7.786	7.798	7.810	7.821	7.833	7.845	7.857	7.869	
9	7.881	7.892	7.904	7.916	7.928	7.940	7.952	7.964	7.976	7.988	

$\pi^3 = 31.0063 \qquad 1/\pi^3 = 0.0322515+$

CUBES *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
2.00	8.000	8.012	8.024	8.036	8.048	8.060	8.072	8.084	8.096	8.108	12
1	8.121	8.133	8.145	8.157	8.169	8.181	8.194	8.206	8.218	8.230	
2	8.242	8.255	8.267	8.279	8.291	8.304	8.316	8.328	8.341	8.353	
3	8.365	8.378	8.390	8.403	8.415	8.427	8.440	8.452	8.465	8.477	
4	8.490	8.502	8.515	8.527	8.540	8.552	8.565	8.577	8.590	8.603	
2.05	8.615	8.628	8.640	8.653	8.666	8.678	8.691	8.704	8.716	8.729	13
6	8.742	8.755	8.767	8.780	8.793	8.806	8.818	8.831	8.844	8.857	
7	8.870	8.883	8.895	8.908	8.921	8.934	8.947	8.960	8.973	8.986	
8	8.999	9.012	9.025	9.038	9.051	9.064	9.077	9.090	9.103	9.116	
9	9.129	9.142	9.156	9.169	9.182	9.195	9.208	9.221	9.235	9.248	
2.10	9.261	9.274	9.287	9.301	9.314	9.327	9.341	9.354	9.367	9.381	
1	9.394	9.407	9.421	9.434	9.447	9.461	9.474	9.488	9.501	9.515	
2	9.528	9.542	9.555	9.569	9.582	9.596	9.609	9.623	9.636	9.650	14
3	9.664	9.677	9.691	9.704	9.718	9.732	9.745	9.759	9.773	9.787	
4	9.800	9.814	9.828	9.842	9.855	9.869	9.883	9.897	9.911	9.925	
2.15	9.938	9.952	9.966	9.980	9.994	10.008					14
2.1						9.94	10.08	10.22	10.36	10.50	14
2	10.65	10.79	10.94	11.09	11.24	11.39	11.54	11.70	11.85	12.01	15
3	12.17	12.33	12.49	12.65	12.81	12.98	13.14	13.31	13.48	13.65	16
4	13.82	14.00	14.17	14.35	14.53	14.71	14.89	15.07	15.25	15.44	18
2.5	15.62	15.81	16.00	16.19	16.39	16.58	16.78	16.97	17.17	17.37	20
6	17.58	17.78	17.98	18.19	18.40	18.61	18.82	19.03	19.25	19.47	21
7	19.68	19.90	20.12	20.35	20.57	20.80	21.02	21.25	21.48	21.72	23
8	21.95	22.19	22.43	22.67	22.91	23.15	23.39	23.64	23.89	24.14	24
9	24.39	24.64	24.90	25.15	25.41	25.67	25.93	26.20	26.46	26.73	26
3.0	27.00	27.27	27.54	27.82	28.09	28.37	28.65	28.93	29.22	29.50	28
1	29.79	30.08	30.37	30.66	30.96	31.26	31.55	31.86	32.16	32.46	30
2	32.77	33.08	33.39	33.70	34.01	34.33	34.65	34.97	35.29	35.61	32
3	35.94	36.26	36.59	36.93	37.26	37.60	37.93	38.27	38.61	38.96	34
4	39.30	39.65	40.00	40.35	40.71	41.06	41.42	41.78	42.14	42.51	36
3.5	42.88	43.24	43.61	43.99	44.36	44.74	45.12	45.50	45.88	46.27	39
6	46.66	47.05	47.44	47.83	48.23	48.63	49.03	49.43	49.84	50.24	40
7	50.65	51.06	51.48	51.90	52.31	52.73	53.16	53.58	54.01	54.44	42
8	54.87	55.31	55.74	56.18	56.62	57.07	57.51	57.96	58.41	58.86	44
9	59.32	59.78	60.24	60.70	61.16	61.63	62.10	62.57	63.04	63.52	47
4.0	64.00	64.48	64.96	65.45	65.94	66.43	66.92	67.42	67.92	68.42	49
1	68.92	69.43	69.93	70.44	70.96	71.47	71.99	72.51	73.03	73.56	52
2	74.09	74.62	75.15	75.69	76.23	76.77	77.31	77.85	78.40	78.95	54
3	79.51	80.06	80.62	81.18	81.75	82.31	82.88	83.45	84.03	84.60	58
4	85.18	85.77	86.35	86.94	87.53	88.12	88.72	89.31	89.92	90.52	59
4.5	91.12	91.73	92.35	92.96	93.58	94.20	94.82	95.44	96.07	96.70	62
6	97.34	97.97	98.61	99.25	99.90	100.54					64
6						100.5	101.2	101.8	102.5	103.2	7
7	103.8	104.5	105.2	105.8	106.5	107.2	107.9	108.5	109.2	109.9	7
8	110.6	111.3	112.0	112.7	113.4	114.1	114.8	115.5	116.2	116.9	7
9	117.6	118.4	119.1	119.8	120.6	121.3	122.0	122.8	123.5	124.3	7

Explanation of Table of Cubes (pp. 8–11).

This table gives the value of N^3 for values of N from 1 to 10, correct to four figures. (Interpolated values may be in error by 1 in the fourth figure.)

To find the cube of a number N outside the range from 1 to 10, note that moving the decimal point **one** place in column N is equivalent to moving it **three** places in the body of the table. For example:

$$(4.852)^3 = 114.2; \quad (0.4852)^3 = 0.1142; \quad (485.2)^3 = 114200000$$

This table may also be used inversely, to give cube roots.

CUBES *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	125.0	125.8	126.5	127.3	128.0	128.8	129.6	130.3	131.1	131.9	8
1	132.7	133.4	134.2	135.0	135.8	136.6	137.4	138.2	139.0	139.8	
2	140.6	141.4	142.2	143.1	143.9	144.7	145.5	146.4	147.2	148.0	
3	148.9	149.7	150.6	151.4	152.3	153.1	154.0	154.9	155.7	156.6	9
4	157.5	158.3	159.2	160.1	161.0	161.9	162.8	163.7	164.6	165.5	
5.5	166.4	167.3	168.2	169.1	170.0	171.0	171.9	172.8	173.7	174.7	
6	175.6	176.6	177.5	178.5	179.4	180.4	181.3	182.3	183.3	184.2	10
7	185.2	186.2	187.1	188.1	189.1	190.1	191.1	192.1	193.1	194.1	
8	195.1	196.1	197.1	198.2	199.2	200.2	201.2	202.3	203.3	204.3	
9	205.4	206.4	207.5	208.5	209.6	210.6	211.7	212.8	213.8	214.9	
6.0	216.0	217.1	218.2	219.3	220.3	221.4	222.5	223.6	224.8	225.9	11
1	227.0	228.1	229.2	230.3	231.5	232.6	233.7	234.9	236.0	237.2	
2	238.3	239.5	240.6	241.8	243.0	244.1	245.3	246.5	247.7	248.9	12
3	250.0	251.2	252.4	253.6	254.8	256.0	257.3	258.5	259.7	260.9	
4	262.1	263.4	264.6	265.8	267.1	268.3	269.6	270.8	272.1	273.4	
6.5	274.6	275.9	277.2	278.4	279.7	281.0	282.3	283.6	284.9	286.2	13
6	287.5	288.8	290.1	291.4	292.8	294.1	295.4	296.7	298.1	299.4	
7	300.8	302.1	303.5	304.8	306.2	307.5	308.9	310.3	311.7	313.0	14
8	314.4	315.8	317.2	318.6	320.0	321.4	322.8	324.2	325.7	327.1	
9	328.5	329.9	331.4	332.8	334.3	335.7	337.2	338.6	340.1	341.5	
7.0	343.0	344.5	345.9	347.4	348.9	350.4	351.9	353.4	354.9	356.4	15
1	357.9	359.4	360.9	362.5	364.0	365.5	367.1	368.6	370.1	371.7	
2	373.2	374.8	376.4	377.9	379.5	381.1	382.7	384.2	385.8	387.4	16
3	389.0	390.6	392.2	393.8	395.4	397.1	398.7	400.3	401.9	403.6	
4	405.2	406.9	408.5	410.2	411.8	413.5	415.2	416.8	418.5	420.2	17
7.5	421.9	423.6	425.3	427.0	428.7	430.4	432.1	433.8	435.5	437.2	
6	439.0	440.7	442.5	444.2	445.9	447.7	449.5	451.2	453.0	454.8	18
7	456.5	458.3	460.1	461.9	463.7	465.5	467.3	469.1	470.9	472.7	
8	474.6	476.4	478.2	480.0	481.9	483.7	485.6	487.4	489.3	491.2	
9	493.0	494.9	496.8	498.7	500.6	502.5	504.4	506.3	508.2	510.1	19
8.0	512.0	513.9	515.8	517.8	519.7	521.7	523.6	525.6	527.5	529.5	
1	531.4	533.4	535.4	537.4	539.4	541.3	543.3	545.3	547.3	549.4	20
2	551.4	553.4	555.4	557.4	559.5	561.5	563.6	565.6	567.7	569.7	
3	571.8	573.9	575.9	578.0	580.1	582.2	584.3	586.4	588.5	590.6	21
4	592.7	594.8	596.9	599.1	601.2	603.4	605.5	607.6	609.8	612.0	
8.5	614.1	616.3	618.5	620.7	622.8	625.0	627.2	629.4	631.6	633.8	22
6	636.1	638.3	640.5	642.7	645.0	647.2	649.5	651.7	654.0	656.2	
7	658.5	660.8	663.1	665.3	667.6	669.9	672.2	674.5	676.8	679.2	23
8	681.5	683.8	686.1	688.5	690.8	693.2	695.5	697.9	700.2	702.6	24
9	705.0	707.3	709.7	712.1	714.5	716.9	719.3	721.7	724.2	726.6	
9.0	729.0	731.4	733.9	736.3	738.8	741.2	743.7	746.1	748.6	751.1	25
1	753.6	756.1	758.6	761.0	763.6	766.1	768.6	771.1	773.6	776.2	
2	778.7	781.2	783.8	786.3	788.9	791.5	794.0	796.6	799.2	801.8	26
3	804.4	807.0	809.6	812.2	814.8	817.4	820.0	822.7	825.3	827.9	
4	830.6	833.2	835.9	838.6	841.2	843.9	846.6	849.3	852.0	854.7	27
9.5	857.4	860.1	862.8	865.5	868.3	871.0	873.7	876.5	879.2	882.0	
6	884.7	887.5	890.3	893.1	895.8	898.6	901.4	904.2	907.0	909.9	28
7	912.7	915.5	918.3	921.2	924.0	926.9	929.7	932.6	935.4	938.3	
8	941.2	944.1	947.0	949.9	952.8	955.7	958.6	961.5	964.4	967.4	29
9	970.3	973.2	976.2	979.1	982.1	985.1	988.0	991.0	994.0	997.0	
10.0	1000.0										

$\pi^3 = 31.0063$ $1/\pi^3 = 0.0322515+$

Moving the decimal point ONE place in N requires moving it THREE places in body of table (see p. 10).

SQUARE ROOTS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044	5
1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091	
2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136	4
3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179	
4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221	
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261	
6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300	
7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338	
8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375	
9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411	
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446	
1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480	3
2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513	
3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546	
4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578	
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609	
6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640	
7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670	
8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700	
9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729	
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758	
1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786	
2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814	
3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841	
4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868	
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895	
6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921	
7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947	
8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972	
9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997	
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022	
1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047	2
2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071	
3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095	
4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119	
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142	
6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166	
7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189	
8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211	
9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234	

$\sqrt{\pi} = 1.77245+$ $1/\sqrt{\pi} = 0.56419$ $\sqrt{\pi/2} = 1.25331$ $\sqrt{e} = 1.64872$

Explanation of Table of Square Roots (pp. 12–15).

This table gives the values of $\sqrt{N}$ for values of N from 1 to 100, correct to four figures. (Interpolated values may be in error by 1 in the fourth figure.)

To find the square root of a number N outside the range from 1 to 100, divide the digits of the number into blocks of two (beginning with the decimal point), and note that moving the decimal point **two** places in N is equivalent to moving it **one** place in the square root of N. For example:

$$\sqrt{2.718} = 1.648; \quad \sqrt{271.8} = 16.48; \quad \sqrt{0.0002718} = 0.01648;$$
$$\sqrt{27.18} = 5.213; \quad \sqrt{2718} = 52.13; \quad \sqrt{0.002718} = 0.05213.$$

SQUARE ROOTS (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256	2
1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278	
2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300	
3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322	
4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343	
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364	
6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385	
7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406	
8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427	
9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447	
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468	
1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488	
2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508	
3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528	
4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548	
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567	
6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587	
7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606	
8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625	
9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644	
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663	
1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681	
2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700	
3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718	
4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737	
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755	
6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773	
7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791	
8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809	
9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827	
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844	
1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862	
2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879	
3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897	
4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914	
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931	
6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948	
7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965	
8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982	
9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998	
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015	
1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032	
2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048	
3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064	
4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081	
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097	
6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113	
7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129	
8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145	
9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161	

Moving the decimal point TWO places in *N* requires moving it ONE place in body of table (see p. 12).

SQUARE ROOTS (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
10.	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302	16
1.	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450	15
2.	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592	14
3.	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728	
4.	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860	13
15.	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987	
6.	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111	12
7.	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231	
8.	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347	
9.	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461	11
20.	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572	
1.	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680	
2.	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785	
3.	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889	10
4.	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990	
25.	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089	
6	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187	
7.	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282	
8.	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376	9
9.	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468	
30.	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559	
1.	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648	
2.	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736	
3.	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822	
4.	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908	8
35.	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992	
6.	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075	
7.	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156	
8.	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237	
9.	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317	
40.	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395	
1.	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473	
2.	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550	
3.	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626	
4.	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701	
45.	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775	7
6.	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848	
7.	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921	
8.	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993	
9.	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064	

SQUARE ROOTS OF CERTAIN FRACTIONS

N	$\sqrt{N}$	*N*	$\sqrt{N}$	*N*	$\sqrt{N}$	*N*	$\sqrt{N}$	*N*	$\sqrt{N}$	*N*	$\sqrt{N}$
1/2	0.7071	3/5	0.7746	4/7	0.7559	1/9	0.3333	5/12	0.6455	9/16	0.7500
1/3	0.5774	4/5	0.8944	5/7	0.8452	2/9	0.4714	7/12	0.7638	11/16	0.8292
2/3	0.8165	1/6	0.4082	6/7	0.9258	4/9	0.6667	11/12	0.9574	13/16	0.9014
1/4	0.5000	5/6	0.9129	1/8	0.3536	5/9	0.7454	1/16	0.2500	15/16	0.9682
3/4	0.8660	1/7	0.3780	3/8	0.6124	7/9	0.8819	3/16	0.4330	1/32	0.1768
1/5	0.4472	2/7	0.5345	5/8	0.7906	8/9	0.9428	5/16	0.5590	1/64	0.1250
2/5	0.6325	3/7	0.6547	7/8	0.9354	1/12	0.2887	7/16	0.6614	1/50	0.1414

SQUARE ROOTS *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
50.	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134	7
1.	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204	
2.	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273	
3.	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342	
4.	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409	
55.	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477	
6.	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543	
7.	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609	
8.	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675	
9.	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740	6
60.	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804	
1.	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868	
2.	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931	
3.	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994	
4.	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056	
65.	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118	
6.	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179	
7.	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240	
8.	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301	
9.	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361	
70.	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420	
1.	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479	
2.	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538	
3.	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597	
4.	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654	
75.	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712	
6.	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769	
7.	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826	
8.	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883	
9.	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939	
80.	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994	
1.	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050	
2.	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105	5
3.	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160	
4.	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214	
85.	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268	
6.	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322	
7.	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375	
8.	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429	
9.	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482	
90.	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534	
1.	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586	
2.	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638	
3.	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690	
4.	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742	
95.	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793	
6.	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844	
7.	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894	
8.	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945	
9.	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995	

$\sqrt{\pi} = 1.77245+$ $\quad 1/\sqrt{\pi} = 0.56419$ $\quad \sqrt{\pi/2} = 1.25331$ $\quad \sqrt{e} = 1.64872$

Moving the decimal point TWO places in N requires moving it ONE place in body of table (see p. 12).

CUBE ROOTS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	1.000	1.003	1.007	1.010	1.013	1.016	1.020	1.023	1.026	1.029	3
1	1.032	1.035	1.038	1.042	1.045	1.048	1.051	1.054	1.057	1.060	
2	1.063	1.066	1.069	1.071	1.074	1.077	1.080	1.083	1.086	1.089	
3	1.091	1.094	1.097	1.100	1.102	1.105	1.108	1.111	1.113	1.116	
4	1.119	1.121	1.124	1.127	1.129	1.132	1.134	1.137	1.140	1.142	
1.5	1.145	1.147	1.150	1.152	1.155	1.157	1.160	1.162	1.165	1.167	2
6	1.170	1.172	1.174	1.177	1.179	1.182	1.184	1.186	1.189	1.191	
7	1.193	1.196	1.198	1.200	1.203	1.205	1.207	1.210	1.212	1.214	
8	1.216	1.219	1.221	1.223	1.225	1.228	1.230	1.232	1.234	1.236	
9	1.239	1.241	1.243	1.245	1.247	1.249	1.251	1.254	1.256	1.258	
2.0	1.260	1.262	1.264	1.266	1.268	1.270	1.272	1.274	1.277	1.279	
1	1.281	1.283	1.285	1.287	1.289	1.291	1.293	1.295	1.297	1.299	
2	1.301	1.303	1.305	1.306	1.308	1.310	1.312	1.314	1.316	1.318	
3	1.320	1.322	1.324	1.326	1.328	1.330	1.331	1.333	1.335	1.337	
4	1.339	1.341	1.343	1.344	1.346	1.348	1.350	1.352	1.354	1.355	
2.5	1.357	1.359	1.361	1.363	1.364	1.366	1.368	1.370	1.372	1.373	
6	1.375	1.377	1.379	1.380	1.382	1.384	1.386	1.387	1.389	1.391	
7	1.392	1.394	1.396	1.398	1.399	1.401	1.403	1.404	1.406	1.408	
8	1.409	1.411	1.413	1.414	1.416	1.418	1.419	1.421	1.423	1.424	
9	1.426	1.428	1.429	1.431	1.433	1.434	1.436	1.437	1.439	1.441	
3.0	1.442	1.444	1.445	1.447	1.449	1.450	1.452	1.453	1.455	1.457	
1	1.458	1.460	1.461	1.463	1.464	1.466	1.467	1.469	1.471	1.472	
2	1.474	1.475	1.477	1.478	1.480	1.481	1.483	1.484	1.486	1.487	
3	1.489	1.490	1.492	1.493	1.495	1.496	1.498	1.499	1.501	1.502	
4	1.504	1.505	1.507	1.508	1.510	1.511	1.512	1.514	1.515	1.517	
3.5	1.518	1.520	1.521	1.523	1.524	1.525	1.527	1.528	1.530	1.531	
6	1.533	1.534	1.535	1.537	1.538	1.540	1.541	1.542	1.544	1.545	1
7	1.547	1.548	1.549	1.551	1.552	1.554	1.555	1.556	1.558	1.559	
8	1.560	1.562	1.563	1.565	1.566	1.567	1.569	1.570	1.571	1.573	
9	1.574	1.575	1.577	1.578	1.579	1.581	1.582	1.583	1.585	1.586	
4.0	1.587	1.589	1.590	1.591	1.593	1.594	1.595	1.597	1.598	1.599	
1	1.601	1.602	1.603	1.604	1.606	1.607	1.608	1.610	1.611	1.612	
2	1.613	1.615	1.616	1.617	1.619	1.620	1.621	1.622	1.624	1.625	
3	1.626	1.627	1.629	1.630	1.631	1.632	1.634	1.635	1.636	1.637	
4	1.639	1.640	1.641	1.642	1.644	1.645	1.646	1.647	1.649	1.650	
4.5	1.651	1.652	1.653	1.655	1.656	1.657	1.658	1.659	1.661	1.662	
6	1.663	1.664	1.666	1.667	1.668	1.669	1.670	1.671	1.673	1.674	
7	1.675	1.676	1.677	1.679	1.680	1.681	1.682	1.683	1.685	1.686	
8	1.687	1.688	1.689	1.690	1.692	1.693	1.694	1.695	1.696	1.697	
9	1.698	1.700	1.701	1.702	1.703	1.704	1.705	1.707	1.708	1.709	

$\sqrt[3]{\pi} = 1.46459 \qquad 1/\sqrt[3]{\pi} = 0.682784$

Explanation of Table of Cube Roots (pp. 16–21).

This table gives the values of $\sqrt[3]{N}$ for all values of N from 1 to 1000, correct to four figures. (Interpolated values may be in error by 1 in the fourth figure.)

To find the cube root of a number N outside the range from 1 to 1000, divide the digits of the number into blocks of three (beginning with the decimal point), and note that moving the decimal point **three** places in column N is equivalent to moving it **one** place in the cube root of N. For example:

$$\sqrt[3]{2.718} = 1.396; \quad \sqrt[3]{2718} = 13.96; \quad \sqrt[3]{0.000002718} = 0.01396.$$
$$\sqrt[3]{27.18} = 3.007; \quad \sqrt[3]{27180} = 30.07; \quad \sqrt[3]{0.00002718} = 0.03007.$$
$$\sqrt[3]{271.8} = 6.477; \quad \sqrt[3]{271800} = 64.77; \quad \sqrt[3]{0.0002718} = 0.06477.$$

CUBE ROOTS (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	1.710	1.711	1.712	1.713	1.715	1.716	1.717	1.718	1.719	1.720	1
1	1.721	1.722	1.724	1.725	1.726	1.727	1.728	1.729	1.730	1.731	
2	1.732	1.734	1.735	1.736	1.737	1.738	1.739	1.740	1.741	1.742	
3	1.744	1.745	1.746	1.747	1.748	1.749	1.750	1.751	1.752	1.753	
4	1.754	1.755	1.757	1.758	1.759	1.760	1.761	1.762	1.763	1.764	
5.5	1.765	1.766	1.767	1.768	1.769	1.771	1.772	1.773	1.774	1.775	
6	1.776	1.777	1.778	1.779	1.780	1.781	1.782	1.783	1.784	1.785	
7	1.786	1.787	1.788	1.789	1.790	1.792	1.793	1.794	1.795	1.796	
8	1.797	1.798	1.799	1.800	1.801	1.802	1.803	1.804	1.805	1.806	
9	1.807	1.808	1.809	1.810	1.811	1.812	1.813	1.814	1.815	1.816	
6.0	1.817	1.818	1.819	1.820	1.821	1.822	1.823	1.824	1.825	1.826	
1	1.827	1.828	1.829	1.830	1.831	1.832	1.833	1.834	1.835	1.836	
2	1.837	1.838	1.839	1.840	1.841	1.842	1.843	1.844	1.845	1.846	
3	1.847	1.848	1.849	1.850	1.851	1.852	1.853	1.854	1.855	1.856	
4	1.857	1.858	1.859	1.860	1.860	1.861	1.862	1.863	1.864	1.865	
6.5	1.866	1.867	1.868	1.869	1.870	1.871	1.872	1.873	1.874	1.875	
6	1.876	1.877	1.878	1.879	1.880	1.881	1.881	1.882	1.883	1.884	
7	1.885	1.886	1.887	1.888	1.889	1.890	1.891	1.892	1.893	1.894	
8	1.895	1.895	1.896	1.897	1.898	1.899	1.900	1.901	1.902	1.903	
9	1.904	1.905	1.906	1.907	1.907	1.908	1.909	1.910	1.911	1.912	
7.0	1.913	1.914	1.915	1.916	1.917	1.917	1.918	1.919	1.920	1.921	
1	1.922	1.923	1.924	1.925	1.926	1.926	1.927	1.928	1.929	1.930	
2	1.931	1.932	1.933	1.934	1.935	1.935	1.936	1.937	1.938	1.939	
3	1.940	1.941	1.942	1.943	1.943	1.944	1.945	1.946	1.947	1.948	
4	1.949	1.950	1.950	1.951	1.952	1.953	1.954	1.955	1.956	1.957	
7.5	1.957	1.958	1.959	1.960	1.961	1.962	1.963	1.964	1.964	1.965	
6	1.966	1.967	1.968	1.969	1.970	1.970	1.971	1.972	1.973	1.974	
7	1.975	1.976	1.976	1.977	1.978	1.979	1.980	1.981	1.981	1.982	
8	1.983	1.984	1.985	1.986	1.987	1.987	1.988	1.989	1.990	1.991	
9	1.992	1.992	1.993	1.994	1.995	1.996	1.997	1.997	1.998	1.999	
8.0	2.000	2.001	2.002	2.002	2.003	2.004	2.005	2.006	2.007	2.007	
1	2.008	2.009	2.010	2.011	2.012	2.012	2.013	2.014	2.015	2.016	
2	2.017	2.017	2.018	2.019	2.020	2.021	2.021	2.022	2.023	2.024	
3	2.025	2.026	2.026	2.027	2.028	2.029	2.030	2.030	2.031	2.032	
4	2.033	2.034	2.034	2.035	2.036	2.037	2.038	2.038	2.039	2.040	
8.5	2.041	2.042	2.042	2.043	2.044	2.045	2.046	2.046	2.047	2.048	
6	2.049	2.050	2.050	2.051	2.052	2.053	2.054	2.054	2.055	2.056	
7	2.057	2.057	2.058	2.059	2.060	2.061	2.061	2.062	2.063	2.064	
8	2.065	2.065	2.066	2.067	2.068	2.068	2.069	2.070	2.071	2.072	
9	2.072	2.073	2.074	2.075	2.075	2.076	2.077	2.078	2.079	2.079	
9.0	2.080	2.081	2.082	2.082	2.083	2.084	2.085	2.085	2.086	2.087	
1	2.088	2.089	2.089	2.090	2.091	2.092	2.092	2.093	2.094	2.095	
2	2.095	2.096	2.097	2.098	2.098	2.099	2.100	2.101	2.101	2.102	
3	2.103	2.104	2.104	2.105	2.106	2.107	2.107	2.108	2.109	2.110	
4	2.110	2.111	2.112	2.113	2.113	2.114	2.115	2.116	2.116	2.117	
9.5	2.118	2.119	2.119	2.120	2.121	2.122	2.122	2.123	2.124	2.125	
6	2.125	2.126	2.127	2.128	2.128	2.129	2.130	2.130	2.131	2.132	
7	2.133	2.133	2.134	2.135	2.136	2.136	2.137	2.138	2.139	2.139	
8	2.140	2.141	2.141	2.142	2.143	2.144	2.144	2.145	2.146	2.147	
9	2.147	2.148	2.149	2.149	2.150	2.151	2.152	2.152	2.153	2.154	

Moving the decimal point THREE places in *N* requires moving it ONE place in body of table (see p. 16).

CUBE ROOTS (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
10.	2.154	2.162	2.169	2.176	2.183	2.190	2.197	2.204	2.210	2.217	7
1.	2.224	2.231	2.237	2.244	2.251	2.257	2.264	2.270	2.277	2.283	6
2.	2.289	2.296	2.302	2.308	2.315	2.321	2.327	2.333	2.339	2.345	
3.	2.351	2.357	2.363	2.369	2.375	2.381	2.387	2.393	2.399	2.404	
4.	2.410	2.416	2.422	2.427	2.433	2.438	2.444	2.450	2.455	2.461	
15.	2.466	2.472	2.477	2.483	2.488	2.493	2.499	2.504	2.509	2.515	5
6.	2.520	2.525	2.530	2.535	2.541	2.546	2.551	2.556	2.561	2.566	
7.	2.571	2.576	2.581	2.586	2.591	2.596	2.601	2.606	2.611	2.616	
8.	2.621	2.626	2.630	2.635	2.640	2.645	2.650	2.654	2.659	2.664	
9.	2.668	2.673	2.678	2.682	2.687	2.692	2.696	2.701	2.705	2.710	
20.	2.714	2.719	2.723	2.728	2.732	2.737	2.741	2.746	2.750	2.755	4
1.	2.759	2.763	2.768	2.772	2.776	2.781	2.785	2.789	2.794	2.798	
2.	2.802	2.806	2.811	2.815	2.819	2.823	2.827	2.831	2.836	2.840	
3.	2.844	2.848	2.852	2.856	2.860	2.864	2.868	2.872	2.876	2.880	
4.	2.884	2.888	2.892	2.896	2.900	2.904	2.908	2.912	2.916	2.920	
25.	2.924	2.928	2.932	2.936	2.940	2.943	2.947	2.951	2.955	2.959	
6.	2.962	2.966	2.970	2.974	2.978	2.981	2.985	2.989	2.993	2.996	
7.	3.000	3.004	3.007	3.011	3.015	3.018	3.022	3.026	3.029	3.033	
8.	3.037	3.040	3.044	3.047	3.051	3.055	3.058	3.062	3.065	3.069	
9.	3.072	3.076	3.079	3.083	3.086	3.090	3.093	3.097	3.100	3.104	
30.	3.107	3.111	3.114	3.118	3.121	3.124	3.128	3.131	3.135	3.138	3
1.	3.141	3.145	3.148	3.151	3.155	3.158	3.162	3.165	3.168	3.171	
2.	3.175	3.178	3.181	3.185	3.188	3.191	3.195	3.198	3.201	3.204	
3.	3.208	3.211	3.214	3.217	3.220	3.224	3.227	3.230	3.233	3.236	
4.	3.240	3.243	3.246	3.249	3.252	3.255	3.259	3.262	3.265	3.268	
35.	3.271	3.274	3.277	3.280	3.283	3.287	3.290	3.293	3.296	3.299	
6.	3.302	3.305	3.308	3.311	3.314	3.317	3.320	3.323	3.326	3.329	
7.	3.332	3.335	3.338	3.341	3.344	3.347	3.350	3.353	3.356	3.359	
8.	3.362	3.365	3.368	3.371	3.374	3.377	3.380	3.382	3.385	3.388	
9.	3.391	3.394	3.397	3.400	3.403	3.406	3.409	3.411	3.414	3.417	
40.	3.420	3.423	3.426	3.428	3.431	3.434	3.437	3.440	3.443	3.445	
1.	3.448	3.451	3.454	3.457	3.459	3.462	3.465	3.468	3.471	3.473	
2.	3.476	3.479	3.482	3.484	3.487	3.490	3.493	3.495	3.498	3.501	
3.	3.503	3.506	3.509	3.512	3.514	3.517	3.520	3.522	3.525	3.528	
4.	3.530	3.533	3.536	3.538	3.541	3.544	3.546	3.549	3.552	3.554	
45.	3.557	3.560	3.562	3.565	3.567	3.570	3.573	3.575	3.578	3.580	
6.	3.583	3.586	3.588	3.591	3.593	3.596	3.599	3.601	3.604	3.606	
7.	3.609	3.611	3.614	3.616	3.619	3.622	3.624	3.627	3.629	3.632	
8.	3.634	3.637	3.639	3.642	3.644	3.647	3.649	3.652	3.654	3.657	2
9.	3.659	3.662	3.664	3.667	3.669	3.672	3.674	3.677	3.679	3.682	

CUBE ROOTS OF CERTAIN FRACTIONS

N	$\sqrt[3]{N}$	N	$\sqrt[3]{N}$	N	$\sqrt[3]{N}$	N	$\sqrt[3]{N}$	N	$\sqrt[3]{N}$	N	$\sqrt[3]{N}$
$1/2$	.7937	$3/5$	.8434	$4/7$	.8298	$1/9$	.4807	$5/12$	.7469	$9/16$	.8255
$1/3$	.6934	$4/5$	.9283	$5/7$	.8939	$2/9$	.6057	$7/12$	.8355	$11/16$	.8826
$2/3$	.8736	$1/6$	.5503	$6/7$	.9499	$4/9$	.7631	$11/12$	.9714	$13/16$	.9331
$1/4$	.6300	$5/6$	.9410	$1/8$	.5000	$5/9$	.8221	$1/16$	.3969	$15/16$	.9787
$3/4$	.9086	$1/7$	.5228	$3/8$	.7211	$7/9$	.9196	$3/16$	.5724	$1/32$	.3150
$1/5$	.5848	$2/7$	.6586	$5/8$	.8550	$8/9$	.9615	$5/16$	.6786	$1/64$	.2500
$2/5$	.7368	$3/7$	.7539	$7/8$	.9565	$1/12$	.4368	$7/16$	.7591	$1/50$	.2714

CUBE ROOTS *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
50.	3.684	3.686	3.689	3.691	3.694	3.696	3.699	3.701	3.704	3.706	2
1.	3.708	3.711	3.713	3.716	3.718	3.721	3.723	3.725	3.728	3.730	
2.	3.733	3.735	3.737	3.740	3.742	3.744	3.747	3.749	3.752	3.754	
3.	3.756	3.759	3.761	3.763	3.766	3.768	3.770	3.773	3.775	3.777	
4.	3.780	3.782	3.784	3.787	3.789	3.791	3.794	3.796	3.798	3.801	
55.	3.803	3.805	3.808	3.810	3.812	3.814	3.817	3.819	3.821	3.824	
6.	3.826	3.828	3.830	3.833	3.835	3.837	3.839	3.842	3.844	3.846	
7.	3.849	3.851	3.853	3.855	3.857	3.860	3.862	3.864	3.866	3.869	
8.	3.871	3.873	3.875	3.878	3.880	3.882	3.884	3.886	3.889	3.891	
9.	3.893	3.895	3.897	3.900	3.902	3.904	3.906	3.908	3.911	3.913	
60.	3.915	3.917	3.919	3.921	3.924	3.926	3.928	3.930	3.932	3.934	
1.	3.936	3.939	3.941	3.943	3.945	3.947	3.949	3.951	3.954	3.956	
2.	3.958	3.960	3.962	3.964	3.966	3.968	3.971	3.973	3.975	3.977	
3.	3.979	3.981	3.983	3.985	3.987	3.990	3.992	3.994	3.996	3.998	
4.	4.000	4.002	4.004	4.006	4.008	4.010	4.012	4.015	4.017	4.019	
65.	4.021	4.023	4.025	4.027	4.029	4.031	4.033	4.035	4.037	4.039	
6.	4.041	4.043	4.045	4.047	4.049	4.051	4.053	4.055	4.058	4.060	
7.	4.062	4.064	4.066	4.068	4.070	4.072	4.074	4.076	4.078	4.080	
8.	4.082	4.084	4.086	4.088	4.090	4.092	4.094	4.096	4.098	4.100	
9.	4.102	4.104	4.106	4.108	4.109	4.111	4.113	4.115	4.117	4.119	
70.	4.121	4.123	4.125	4.127	4.129	4.131	4.133	4.135	4.137	4.139	
1.	4.141	4.143	4.145	4.147	4.149	4.151	4.152	4.154	4.156	4.158	
2.	4.160	4.162	4.164	4.166	4.168	4.170	4.172	4.174	4.176	4.177	
3.	4.179	4.181	4.183	4.185	4.187	4.189	4.191	4.193	4.195	4.196	
4.	4.198	4.200	4.202	4.204	4.206	4.208	4.210	4.212	4.213	4.215	
75.	4.217	4.219	4.221	4.223	4.225	4.227	4.228	4.230	4.232	4.234	
6.	4.236	4.238	4.240	4.241	4.243	4.245	4.247	4.249	4.251	4.252	
7.	4.254	4.256	4.258	4.260	4.262	4.264	4.265	4.267	4.269	4.271	
8.	4.273	4.274	4.276	4.278	4.280	4.282	4.284	4.285	4.287	4.289	
9.	4.291	4.293	4.294	4.296	4.298	4.300	4.302	4.303	4.305	4.307	
80.	4.309	4.311	4.312	4.314	4.316	4.318	4.320	4.321	4.323	4.325	
1.	4.327	4.329	4.330	4.332	4.334	4.336	4.337	4.339	4.341	4.343	
2.	4.344	4.346	4.348	4.350	4.352	4.353	4.355	4.357	4.359	4.360	
3.	4.362	4.364	4.366	4.367	4.369	4.371	4.373	4.374	4.376	4.378	
4.	4.380	4.381	4.383	4.385	4.386	4.388	4.390	4.392	4.393	4.395	
85.	4.397	4.399	4.400	4.402	4.404	4.405	4.407	4.409	4.411	4.412	
6.	4.414	4.416	4.417	4.419	4.421	4.423	4.424	4.426	4.428	4.429	
7.	4.431	4.433	4.434	4.436	4.438	4.440	4.441	4.443	4.445	4.446	
8.	4.448	4.450	4.451	4.453	4.455	4.456	4.458	4.460	4.461	4.463	
9.	4.465	4.466	4.468	4.470	4.471	4.473	4.475	4.476	4.478	4.480	
90.	4.481	4.483	4.485	4.486	4.488	4.490	4.491	4.493	4.495	4.496	
1.	4.498	4.500	4.501	4.503	4.505	4.506	4.508	4.509	4.511	4.513	
2.	4.514	4.516	4.518	4.519	4.521	4.523	4.524	4.526	4.527	4.529	
3.	4.531	4.532	4.534	4.536	4.537	4.539	4.540	4.542	4.544	4.545	
4.	4.547	4.548	4.550	4.552	4.553	4.555	4.556	4.558	4.560	4.561	
95.	4.563	4.565	4.566	4.568	4.569	4.571	4.572	4.574	4.576	4.577	
6.	4.579	4.580	4.582	4.584	4.585	4.587	4.588	4.590	4.592	4.593	
7.	4.595	4.596	4.598	4.599	4.601	4.603	4.604	4.606	4.607	4.609	
8.	4.610	4.612	4.614	4.615	4.617	4.618	4.620	4.621	4.623	4.625	
9.	4.626	4.628	4.629	4.631	4.632	4.634	4.635	4.637	4.638	4.640	

Moving the decimal point THREE places in *N* requires moving it ONE place in body of table (see p. 16).

CUBE ROOTS *(continued)*

N	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	Avg. diff.
10	4.642	4.657	4.672	4.688	4.703	4.718	4.733	4.747	4.762	4.777	15
1	4.791	4.806	4.820	4.835	4.849	4.863	4.877	4.891	4.905	4.919	14
2	4.932	4.946	4.960	4.973	4.987	5.000	5.013	5.027	5.040	5.053	13
3	5.066	5.079	5.092	5.104	5.117	5.130	5.143	5.155	5.168	5.180	
4	5.192	5.205	5.217	5.229	5.241	5.254	5.266	5.278	5.290	5.301	12
15	5.313	5.325	5.337	5.348	5.360	5.372	5.383	5.395	5.406	5.418	
6	5.429	5.440	5.451	5.463	5.474	5.485	5.496	5.507	5.518	5.529	11
7	5.540	5.550	5.561	5.572	5.583	5.593	5.604	5.615	5.625	5.636	
8	5.646	5.657	5.667	5.677	5.688	5.698	5.708	5.718	5.729	5.739	10
9	5.749	5.759	5.769	5.779	5.789	5.799	5.809	5.819	5.828	5.838	
20	5.848	5.858	5.867	5.877	5.887	5.896	5.906	5.915	5.925	5.934	
1	5.944	5.953	5.963	5.972	5.981	5.991	6.000	6.009	6.018	6.028	9
2	6.037	6.046	6.055	6.064	6.073	6.082	6.091	6.100	6.109	6.118	
3	6.127	6.136	6.145	6.153	6.162	6.171	6.180	6.188	6.197	6.206	
4	6.214	6.223	6.232	6.240	6.249	6.257	6.266	6.274	6.283	6.291	
25	6.300	6.308	6.316	6.325	6.333	6.341	6.350	6.358	6.366	6.374	8
6	6.383	6.391	6.399	6.407	6.415	6.423	6.431	6.439	6.447	6.455	
7	6.463	6.471	6.479	6.487	6.495	6.503	6.511	6.519	6.527	6.534	
8	6.542	6.550	6.558	6.565	6.573	6.581	6.589	6.596	6.604	6.611	
9	6.619	6.627	6.634	6.642	6.649	6.657	6.664	6.672	6.679	6.687	
30	6.694	6.702	6.709	6.717	6.724	6.731	6.739	6.746	6.753	6.761	7
1	6.768	6.775	6.782	6.790	6.797	6.804	6.811	6.818	6.826	6.833	
2	6.840	6.847	6.854	6.861	6.868	6.875	6.882	6.889	6.896	6.903	
3	6.910	6.917	6.924	6.931	6.938	6.945	6.952	6.959	6.966	6.973	
4	6.980	6.986	6.993	7.000	7.007	7.014	7.020	7.027	7.034	7.041	
35	7.047	7.054	7.061	7.067	7.074	7.081	7.087	7.094	7.101	7.107	
6	7.114	7.120	7.127	7.133	7.140	7.147	7.153	7.160	7.166	7.173	6
7	7.179	7.186	7.192	7.198	7.205	7.211	7.218	7.224	7.230	7.237	
8	7.243	7.250	7.256	7.262	7.268	7.275	7.281	7.287	7.294	7.300	
9	7.306	7.312	7.319	7.325	7.331	7.337	7.343	7.350	7.356	7.362	
40	7.368	7.374	7.380	7.386	7.393	7.399	7.405	7.411	7.417	7.423	
1	7.429	7.435	7.441	7.447	7.453	7.459	7.465	7.471	7.477	7.483	
2	7.489	7.495	7.501	7.507	7.513	7.518	7.524	7.530	7.536	7.542	
3	7.548	7.554	7.560	7.565	7.571	7.577	7.583	7.589	7.594	7.600	
4	7.606	7.612	7.617	7.623	7.629	7.635	7.640	7.646	7.652	7.657	
45	7.663	7.669	7.674	7.680	7.686	7.691	7.697	7.703	7.708	7.714	5
6	7.719	7.725	7.731	7.736	7.742	7.747	7.753	7.758	7.764	7.769	
7	7.775	7.780	7.786	7.791	7.797	7.802	7.808	7.813	7.819	7.824	
8	7.830	7.835	7.841	7.846	7.851	7.857	7.862	7.868	7.873	7.878	
9	7.884	7.889	7.894	7.900	7.905	7.910	7.916	7.921	7.926	7.932	

AUXILIARY TABLE OF TWO-THIRDS POWERS AND THREE-HALVES POWERS (see pp. 22–23)

(To assist in locating the decimal point)

N	$N^{2/3}(=\sqrt[3]{N^2})$	$N^{3/2}(=\sqrt{N^3})$
.0001	.002154	.000001
.001	.01	00003162
.01	.0464	.001
.1	.2154	.03162278
1.	1.	1.
10.	4.64	31.62278
100.	21.54	1000.
1000.	100.	31622.78
10000.	464.16	1000000.

For complete table of three-halves powers, see pp. 22–23. That table, used inversely, provides a complete table of two-thirds powers.

CUBE ROOTS (*continued*)

N	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.	Avg. diff.
50	7.937	7.942	7.948	7.953	7.958	7.963	7.969	7.974	7.979	7.984	5
1	7.990	7.995	8.000	8.005	8.010	8.016	8.021	8.026	8.031	8.036	
2	8.041	8.047	8.052	8.057	8.062	8.067	8.072	8.077	8.082	8.088	
3	8.093	8.098	8.103	8.108	8.113	8.118	8.123	8.128	8.133	8.138	
4	8.143	8.148	8.153	8.158	8.163	8.168	8.173	8.178	8.183	8.188	
55	8.193	8.198	8.203	8.208	8.213	8.218	8.223	8.228	8.233	8.238	
6	8.243	8.247	8.252	8.257	8.262	8.267	8.272	8.277	8.282	8.286	
7	8.291	8.296	8.301	8.306	8.311	8.316	8.320	8.325	8.330	8.335	
8	8.340	8.344	8.349	8.354	8.359	8.363	8.368	8.373	8.378	8.382	
9	8.387	8.392	8.397	8.401	8.406	8.411	8.416	8.420	8.425	8.430	
60	8.434	8.439	8.444	8.448	8.453	8.458	8.462	8.467	8.472	8.476	
1	8.481	8.486	8.490	8.495	8.499	8.504	8.509	8.513	8.518	8.522	
2	8.527	8.532	8.536	8.541	8.545	8.550	8.554	8.559	8.564	8.568	
3	8.573	8.577	8.582	8.586	8.591	8.595	8.600	8.604	8.609	8.613	4
4	8.618	8.622	8.627	8.631	8.636	8.640	8.645	8.649	8.653	8.658	
65	8.662	8.667	8.671	8.676	8.680	8.685	8.689	8.693	8.698	8.702	
6	8.707	8.711	8.715	8.720	8.724	8.729	8.733	8.737	8.742	8.746	
7	8.750	8.755	8.759	8.763	8.768	8.772	8.776	8.781	8.785	8.789	
8	8.794	8.798	8.802	8.807	8.811	8.815	8.819	8.824	8.828	8.832	
9	8.837	8.841	8.845	8.849	8.854	8.858	8.862	8.866	8.871	8.875	
70	8.879	8.883	8.887	8.892	8.896	8.900	8.904	8.909	8.913	8.917	
1	8.921	8.925	8.929	8.934	8.938	8.942	8.946	8.950	8.955	8.959	
2	8.963	8.967	8.971	8.975	8.979	8.984	8.988	8.992	8.996	9.000	
3	9.004	9.008	9.012	9.016	9.021	9.025	9.029	9.033	9.037	9.041	
4	9.045	9.049	9.053	9.057	9.061	9.065	9.069	9.073	9.078	9.082	
75	9.086	9.090	9.094	9.098	9.102	9.106	9.110	9.114	9.118	9.122	
6	9.126	9.130	9.134	9.138	9.142	9.146	9.150	9.154	9.158	9.162	
7	9.166	9.170	9.174	9.178	9.182	9.185	9.189	9.193	9.197	9.201	
8	9.205	9.209	9.213	9.217	9.221	9.225	9.229	9.233	9.237	9.240	
9	9.244	9.248	9.252	9.256	9.260	9.264	9.268	9.272	9.275	9.279	
80	9.283	9.287	9.291	9.295	9.299	9.302	9.306	9.310	9.314	9.318	
1	9.322	9.326	9.329	9.333	9.337	9.341	9.345	9.348	9.352	9.356	
2	9.360	9.364	9.368	9.371	9.375	9.379	9.383	9.386	9.390	9.394	
3	9.398	9.402	9.405	9.409	9.413	9.417	9.420	9.424	9.428	9.432	
4	9.435	9.439	9.443	9.447	9.450	9.454	9.458	9.462	9.465	9.469	
85	9.473	9.476	9.480	9.484	9.488	9.491	9.495	9.499	9.502	9.506	
6	9.510	9.513	9.517	9.521	9.524	9.528	9.532	9.535	9.539	9.543	
7	9.546	9.550	9.554	9.557	9.561	9.565	9.568	9.572	9.576	9.579	
8	9.583	9.586	9.590	9.594	9.597	9.601	9.605	9.608	9.612	9.615	
9	9.619	9.623	9.626	9.630	9.633	9.637	9.641	9.644	9.648	9.651	
90	9.655	9.658	9.662	9.666	9.669	9.673	9.676	9.680	9.683	9.687	
1	9.691	9.694	9.698	9.701	9.705	9.708	9.712	9.715	9.719	9.722	
2	9.726	9.729	9.733	9.736	9.740	9.743	9.747	9.750	9.754	9.758	
3	9.761	9.764	9.768	9.771	9.775	9.778	9.782	9.785	9.789	9.792	
4	9.796	9.799	9.803	9.806	9.810	9.813	9.817	9.820	9.824	9.827	
95	9.830	9.834	9.837	9.841	9.844	9.848	9.851	9.855	9.858	9.861	
6	9.865	9.868	9.872	9.875	9.879	9.882	9.885	9.889	9.892	9.896	
7	9.899	9.902	9.906	9.909	9.913	9.916	9.919	9.923	9.926	9.930	
8	9.933	9.936	9.940	9.943	9.946	9.950	9.953	9.956	9.960	9.963	
9	9.967	9.970	9.973	9.977	9.980	9.983	9.987	9.990	9.993	9.997	
100	10.00										

Moving the decimal point THREE places in *N* requires moving it ONE place in body of table (see p. 16).

THREE-HALVES POWERS OF NUMBERS (see also p. 20)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.	1.000	1.154	1.315	1.482	1.657	1.837	2.024	2.217	2.415	2.619	183
2.	2.828	3.043	3.263	3.488	3.718	3.953	4.192	4.437	4.685	4.939	237
3.	5.196	5.458	5.724	5.995	6.269	6.548	6.831	7.117	7.408	7.702	280
4.	8.000	8.302	8.607	8.917	9.230	9.546	9.866	10.190			313
4.								10.19	10.52	10.85	33
5.	11.18	11.52	11.86	12.20	12.55	12.90	13.25	13.61	13.97	14.33	35
6.	14.70	15.07	15.44	15.81	16.19	16.57	16.96	17.34	17.73	18.12	38
7.	18.52	18.92	19.32	19.72	20.13	20.54	20.95	21.37	21.78	22.20	41
8.	22.63	23.05	23.48	23.91	24.35	24.78	25.22	25.66	26.11	26.55	44
9.	27.00	27.45	27.90	28.36	28.82	29.28	29.74	30.21	30.68	31.15	46
10.	31.62	32.10	32.58	33.06	33.54	34.02	34.51	35.00	35.49	35.99	49
1.	36.48	36.98	37.48	37.99	38.49	39.00	39.51	40.02	40.53	41.05	51
2.	41.57	42.09	42.61	43.14	43.66	44.19	44.73	45.26	45.79	46.33	53
3.	46.87	47.41	47.96	48.50	49.05	49.60	50.15	50.71	51.26	51.82	55
4.	52.38	52.95	53.51	54.08	54.64	55.21	55.79	56.36	56.94	57.51	57
15.	58.09	58.68	59.26	59.85	60.43	61.02	61.62	62.21	62.80	63.40	59
6.	64.00	64.60	65.20	65.81	66.41	67.02	67.63	68.25	68.86	69.48	61
7.	70.09	70.71	71.33	71.96	72.58	73.21	73.84	74.47	75.10	75.73	63
8.	76.37	77.00	77.64	78.28	78.93	79.57	80.22	80.87	81.51	82.17	65
9.	82.82	83.47	84.13	84.79	85.45	86.11	86.77	87.44	88.10	88.77	66
20.	89.44	90.11	90.79	91.46	92.14	92.82	93.50	94.18	94.86	95.55	68
1.	96.23	96.92	97.61	98.30	99.00	99.69	100.38				69
1.							100.4	101.1	101.8	102.5	7
2.	103.2	103.9	104.6	105.3	106.0	106.7	107.4	108.2	108.9	109.6	7
3.	110.3	111.0	111.7	112.5	113.2	113.9	114.6	115.4	116.1	116.8	7
4.	117.6	118.3	119.0	119.8	120.5	121.3	122.0	122.8	123.5	124.3	7
25.	125.0	125.8	126.5	127.3	128.0	128.8	129.5	130.3	131.0	131.8	8
6.	132.6	133.3	134.1	134.9	135.6	136.4	137.2	138.0	138.7	139.5	8
7.	140.3	141.1	141.9	142.6	143.4	144.2	145.0	145.8	146.6	147.4	8
8.	148.2	149.0	149.8	150.5	151.3	152.1	152.9	153.8	154.6	155.4	8
9.	156.2	157.0	157.8	158.6	159.4	160.2	161.0	161.9	162.7	163.5	8
30.	164.3	165.1	166.0	166.8	167.6	168.4	169.3	170.1	170.9	171.8	8
1.	172.6	173.4	174.3	175.1	176.0	176.8	177.6	178.5	179.3	180.2	8
2.	181.0	181.9	182.7	183.6	184.4	185.3	186.1	187.0	187.8	188.7	9
3.	189.6	190.4	191.3	192.2	193.0	193.9	194.8	195.6	196.5	197.4	9
4.	198.3	199.1	200.0	200.9	201.8	202.6	203.5	204.4	205.3	206.2	9
35.	207.1	208.0	208.8	209.7	210.6	211.5	212.4	213.3	214.2	215.1	9
6.	216.0	216.9	217.8	218.7	219.6	220.5	221.4	222.3	223.2	224.2	9
7.	225.1	226.0	226.9	227.8	228.7	229.6	230.6	231.5	232.4	233.3	9
8.	234.2	235.2	236.1	237.0	238.0	238.9	239.8	240.8	241.7	242.6	9
9.	243.6	244.5	245.4	246.4	247.3	248.3	249.2	250.1	251.1	252.0	9
40.	253.0	253.9	254.9	255.8	256.8	257.7	258.7	259.7	260.6	261.6	10
1.	262.5	263.5	264.5	265.4	266.4	267.3	268.3	269.3	270.2	271.2	10
2.	272.2	273.2	274.1	275.1	276.1	277.1	278.0	279.0	280.0	281.0	10
3.	282.0	283.0	283.9	284.9	285.9	286.9	287.9	288.9	289.9	290.9	10
4.	291.9	292.9	293.9	294.9	295.9	296.9	297.9	298.9	299.9	300.9	10
45.	301.9	302.9	303.9	304.9	305.9	306.9	307.9	308.9	310.0	311.0	10
6.	312.0	313.0	314.0	315.0	316.1	317.1	318.1	319.1	320.2	321.2	10
7.	322.2	323.2	324.3	325.3	326.3	327.4	328.4	329.4	330.5	331.5	10
8.	332.6	333.6	334.6	335.7	336.7	337.8	338.8	339.9	340.9	342.0	10
9.	343.0	344.1	345.1	346.2	347.2	348.3	349.3	350.4	351.4	352.5	11

This table gives $N^{3/2}$ from $N = 1$ to $N = 100$. Moving the decimal point TWO places in N requires moving it THREE places in body of table. Thus:

$(7.23)^{3/2} = 19.44$; $(723.)^{3/2} = 19440$; $(0.0723)^{3/2} = 0.01944$

$(72.3)^{3/2} = 614.8$; $(7230.)^{3/2} = 614800$; $(0.723)^{3/2} = 0.6148$

Used inversely, table gives $M^{2/3}$ from $M = 1$ to $M = 1000$. Thus: $(0.6148)^{2/3} = 0.7230$.

THREE-HALVES POWERS (*continued*) (See also p. 20)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
50.	353.6	354.6	355.7	356.7	357.8	358.9	359.9	361.0	362.1	363.1	11
1.	364.2	365.3	366.4	367.4	368.5	369.6	370.7	371.7	372.8	373.9	11
2.	375.0	376.1	377.1	378.2	379.3	380.4	381.5	382.6	383.7	384.8	11
3	385.8	386.9	388.0	389.1	390.2	391.3	392.4	393.5	394.6	395.7	11
4.	396.8	397.9	399.0	400.1	401.2	402.3	403.4	404.6	405.7	406.8	11
55.	407.9	409.0	410.1	411.2	412.3	413.5	414.6	415.7	416.8	417.9	11
6.	419.1	420.2	421.3	422.4	423.6	424.7	425.8	426.9	428.1	429.2	11
7.	430.3	431.5	432.6	433.7	434.9	436.0	437.2	438.3	439.4	440.6	11
8.	441.7	442.9	444.0	445.1	446.3	447.4	448.6	449.7	450.9	452.0	11
9.	453.2	454.3	455.5	456.6	457.8	459.0	460.1	461.3	462.4	463.6	12
60.	464.8	465.9	467.1	468.2	469.4	470.6	471.7	472.9	474.1	475.3	12
1.	476.4	477.6	478.8	479.9	481.1	482.3	483.5	484.6	485.8	487.0	12
2.	488.2	489.4	490.6	491.7	492.9	494.1	495.3	496.5	497.7	498.9	12
3.	500.0	501.2	502.4	503.6	504.8	506.0	507.2	508.4	509.6	510.8	12
4.	512.0	513.2	514.4	515.6	516.8	518.0	519.2	520.4	521.6	522.8	12
65.	524.0	525.3	526.5	527.7	528.9	530.1	531.3	532.5	533.8	535.0	12
6.	536.2	537.4	538.6	539.8	541.1	542.3	543.5	544.7	546.0	547.2	12
7.	548.4	549.6	550.9	552.1	553.3	554.6	555.8	557.0	558.3	559.5	12
8.	560.7	562.0	563.2	564.5	565.7	566.9	568.2	569.4	570.7	571.9	12
9.	573.2	574.4	575.7	576.9	578.1	579.4	580.6	581.9	583.2	584.4	13
70.	585.7	586.9	588.2	589.4	590.7	591.9	593.2	594.5	595.7	597.0	13
1.	598.3	599.5	600.8	602.1	603.3	604.6	605.9	607.1	608.4	609.7	13
2.	610.9	612.2	613.5	614.8	616.0	617.3	618.6	619.9	621.2	622.4	13
3.	623.7	625.0	626.3	627.6	628.8	630.1	631.4	632.7	634.0	635.3	13
4.	636.6	637.9	639.2	640.4	641.7	643.0	644.3	645.6	646.9	648.2	13
75.	649.5	650.8	652.1	653.4	654.7	656.0	657.3	658.6	659.9	661.2	13
6.	662.6	663.9	665.2	666.5	667.8	669.1	670.4	671.7	673.0	674.4	13
7.	675.7	677.0	678.3	679.6	680.9	682.3	683.6	684.9	686.2	687.6	13
8.	688.9	690.2	691.5	692.9	694.2	695.5	696.8	698.2	699.5	700.8	13
9.	702.2	703.5	704.8	706.2	707.5	708.8	710.2	711.5	712.9	714.2	13
80.	715.5	716.9	718.2	719.6	720.9	722.3	723.6	725.0	726.3	727.7	13
1.	729.0	730.4	731.7	733.1	734.4	735.8	737.1	738.5	739.8	741.2	14
2.	742.5	743.9	745.3	746.6	748.0	749.3	750.7	752.1	753.4	754.8	14
3.	756.2	757.5	758.9	760.3	761.6	763.0	764.4	765.8	767.1	768.5	14
4.	769.9	771.2	772.6	774.0	775.4	776.8	778.1	779.5	780.9	782.3	14
85.	783.7	785.0	786.4	787.8	789.2	790.6	792.0	793.4	794.8	796.1	14
6.	797.5	798.9	800.3	801.7	803.1	804.5	805.9	807.3	808.7	810.1	14
7.	811.5	812.9	814.3	815.7	817.1	818.5	819.9	821.3	822.7	824.1	14
8.	825.5	826.9	828.3	829.7	831.1	832.6	834.0	835.4	836.8	838.2	14
9.	839.6	841.0	842.5	843.9	845.3	846.7	848.1	849.5	851.0	852.4	14
90.	853.8	855.2	856.7	858.1	859.5	860.9	862.4	863.8	865.2	866.7	14
1.	868.1	869.5	870.9	872.4	873.8	875.2	876.7	878.1	879.6	881.0	14
2.	882.4	883.9	885.3	886.8	888.2	889.6	891.1	892.5	894.0	895.4	14
3.	896.9	898.3	899.8	901.2	902.7	904.1	905.6	907.0	908.5	909.9	15
4.	911.4	912.8	914.3	915.7	917.2	918.6	920.1	921.6	923.0	924.5	15
95.	925.9	927.4	928.9	930.3	931.8	933.3	934.7	936.2	937.7	939.1	15
6.	940.6	942.1	943.5	945.0	946.5	948.0	949.4	950.9	952.4	953.9	15
7.	955.3	956.8	958.3	959.8	961.3	962.7	964.2	965.7	967.2	968.7	15
8.	970.2	971.6	973.1	974.6	976.1	977.6	979.1	980.6	982.1	983.5	15
9.	985.0	986.5	988.0	989.5	991.0	992.5	994.0	995.5	997.0	998.5	15
100.	1000.0										

Moving the decimal point TWO places in *N* requires moving it THREE places in body of table (see also auxiliary table on p. 20).

RECIPROCALS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.00		.9990	.9980	.9970	.9960	.9950	.9940	.9930	.9921	.9911	−10
1	.9901	.9891	.9881	.9872	.9862	.9852	.9843	.9833	.9823	.9814	
2	.9804	.9794	.9785	.9775	.9766	.9756	.9747	.9737	.9728	.9718	
3	.9709	.9699	.9690	.9681	.9671	.9662	.9653	.9643	.9634	.9625	−9
4	.9615	.9606	.9597	.9588	.9579	.9569	.9560	.9551	.9542	.9533	
1.05	.9524	.9515	.9506	.9497	.9488	.9479	.9470	.9461	.9452	.9443	
6	.9434	.9425	.9416	.9407	.9398	.9390	.9381	.9372	.9363	.9355	
7	.9346	.9337	.9328	.9320	.9311	.9302	.9294	.9285	.9276	.9268	
8	.9259	.9251	.9242	.9234	.9225	.9217	.9208	.9200	.9191	.9183	−8
9	.9174	.9166	.9158	.9149	.9141	.9132	.9124	.9116	.9107	.9099	
1.10	.9091	.9083	.9074	.9066	.9058	.9050	.9042	.9033	.9025	.9017	
1	.9009	.9001	.8993	.8985	.8977	.8969	.8961	.8953	.8945	.8937	
2	.8929	.8921	.8913	.8905	.8897	.8889	.8881	.8873	.8865	.8857	
3	.8850	.8842	.8834	.8826	.8818	.8811	.8803	.8795	.8787	.8780	
4	.8772	.8764	.8757	.8749	.8741	.8734	.8726	.8718	.8711	.8703	
1.15	.8696	.8688	.8681	.8673	.8666	.8658	.8651	.8643	.8636	.8628	
6	.8621	.8613	.8606	.8598	.8591	.8584	.8576	.8569	.8562	.8554	−7
7	.8547	.8540	.8532	.8525	.8518	.8511	.8503	.8496	.8489	.8482	
8	.8475	.8467	.8460	.8453	.8446	.8439	.8432	.8425	.8418	.8410	
9	.8403	.8396	.8389	.8382	.8375	.8368	.8361	.8354	.8347	.8340	
1.20	.8333	.8326	.8319	.8313	.8306	.8299	.8292	.8285	.8278	.8271	
1	.8264	.8258	.8251	.8244	.8237	.8230	.8224	.8217	.8210	.8203	
2	.8197	.8190	.8183	.8177	.8170	.8163	.8157	.8150	.8143	.8137	
3	.8130	.8123	.8117	.8110	.8104	.8097	.8091	.8084	.8078	.8071	−6
4	.8065	.8058	.8052	.8045	.8039	.8032	.8026	.8019	.8013	.8006	
1.25	.8000	.7994	.7987	.7981	.7974	.7968	.7962	.7955	.7949	.7943	
6	.7937	.7930	.7924	.7918	.7911	.7905	.7899	.7893	.7886	.7880	
7	.7874	.7868	.7862	.7855	.7849	.7843	.7837	.7831	.7825	.7819	
8	.7812	.7806	.7800	.7794	.7788	.7782	.7776	.7770	.7764	.7758	
9	.7752	.7746	.7740	.7734	.7728	.7722	.7716	.7710	.7704	.7698	
1.30	.7692	.7686	.7680	.7675	.7669	.7663	.7657	.7651	.7645	.7639	
1	.7634	.7628	.7622	.7616	.7610	.7605	.7599	.7593	.7587	.7582	
2	.7576	.7570	.7564	.7559	.7553	.7547	.7541	.7536	.7530	.7524	
3	.7519	.7513	.7508	.7502	.7496	.7491	.7485	.7479	.7474	.7468	
4	.7463	.7457	.7452	.7446	.7440	.7435	.7429	.7424	.7418	.7413	
1.35	.7407	.7402	.7396	.7391	.7386	.7380	.7375	.7369	.7364	.7358	−5
6	.7353	.7348	.7342	.7337	.7331	.7326	.7321	.7315	.7310	.7305	
7	.7299	.7294	.7289	.7283	.7278	.7273	.7267	.7262	.7257	.7252	
8	.7246	.7241	.7236	.7231	.7225	.7220	.7215	.7210	.7205	.7199	
9	.7194	.7189	.7184	.7179	.7174	.7168	.7163	.7158	.7153	.7148	
1.40	.7143	.7138	.7133	.7128	.7123	.7117	.7112	.7107	.7102	.7097	
1	.7092	.7087	.7082	.7077	.7072	.7067	.7062	.7057	.7052	.7047	
2	.7042	.7037	.7032	.7027	.7022	.7018	.7013	.7008	.7003	.6998	
3	.6993	.6988	.6983	.6978	.6974	.6969	.6964	.6959	.6954	.6949	
4	.6944	.6940	.6935	.6930	.6925	.6920	.6916	.6911	.6906	.6901	
1.45	.6897	.6892	.6887	.6882	.6878	.6873	.6868	.6863	.6859	.6854	
6	.6849	.6845	.6840	.6835	.6831	.6826	.6821	.6817	.6812	.6807	
7	.6803	.6798	.6793	.6789	.6784	.6780	.6775	.6770	.6766	.6761	
8	.6757	.6752	.6748	.6743	.6739	.6734	.6729	.6725	.6720	.6716	
9	.6711	.6707	.6702	.6698	.6693	.6689	.6684	.6680	.6676	.6671	

$1/\pi = 0.318310$ $1/e = 0.367879$

Moving the decimal point in either direction in N requires moving it in the OPPOSITE direction in body of table (see p. 26).

RECIPROCALS *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.50	.6667	.6662	.6658	.6653	.6649	.6645	.6640	.6636	.6631	.6627	−4
1	.6623	.6618	.6614	.6609	.6605	.6601	.6596	.6592	.6588	.6583	
2	.6579	.6575	.6570	.6566	.6562	.6557	.6553	.6549	.6545	.6540	
3	.6536	.6532	.6527	.6523	.6519	.6515	.6510	.6506	.6502	.6498	
4	.6494	.6489	.6485	.6481	.6477	.6472	.6468	.6464	.6460	.6456	
1.55	.6452	.6447	.6443	.6439	.6435	.6431	.6427	.6423	.6418	.6414	
6	.6410	.6406	.6402	.6398	.6394	.6390	.6386	.6382	.6378	.6373	
7	.6369	.6365	.6361	.6357	.6353	.6349	.6345	.6341	.6337	.6333	
8	.6329	.6325	.6321	.6317	.6313	.6309	.6305	.6301	.6297	.6293	
9	.6289	.6285	.6281	.6277	.6274	.6270	.6266	.6262	.6258	.6254	
1.60	.6250	.6246	.6242	.6238	.6234	.6231	.6227	.6223	.6219	.6215	
1	.6211	.6207	.6203	.6200	.6196	.6192	.6188	.6184	.6180	.6177	
2	.6173	.6169	.6165	.6161	.6158	.6154	.6150	.6146	.6143	.6139	
3	.6135	.6131	.6127	.6124	.6120	.6116	.6112	.6109	.6105	.6101	
4	.6098	.6094	.6090	.6086	.6083	.6079	.6075	.6072	.6068	.6064	
1.65	.6061	.6057	.6053	.6050	.6046	.6042	.6039	.6035	.6031	.6028	
6	.6024	.6020	.6017	.6013	.6010	.6006	.6002	.5999	.5995	.5992	
7	.5988	.5984	.5981	.5977	.5974	.5970	.5967	.5963	.5959	.5956	
8	.5952	.5949	.5945	.5942	.5938	.5935	.5931	.5928	.5924	.5921	
9	.5917	.5914	.5910	.5907	.5903	.5900	.5896	.5893	.5889	.5886	
1.70	.5882	.5879	.5875	.5872	.5869	.5865	.5862	.5858	.5855	.5851	−3
1	.5848	.5845	.5841	.5838	.5834	.5831	.5828	.5824	.5821	.5817	
2	.5814	.5811	.5807	.5804	.5800	.5797	.5794	.5790	.5787	.5784	
3	.5780	.5777	.5774	.5770	.5767	.5764	.5760	.5757	.5754	.5750	
4	.5747	.5744	.5741	.5737	.5734	.5731	.5727	.5724	.5721	.5718	
1.75	.5714	.5711	.5708	.5705	.5701	.5698	.5695	.5692	.5688	.5685	
6	.5682	.5679	.5675	.5672	.5669	.5666	.5663	.5659	.5656	.5653	
7	.5650	.5647	.5643	.5640	.5637	.5634	.5631	.5627	.5624	.5621	
8	.5618	.5615	.5612	.5609	.5605	.5602	.5599	.5596	.5593	.5590	
9	.5587	.5583	.5580	.5577	.5574	.5571	.5568	.5565	.5562	.5559	
1.80	.5556	.5552	.5549	.5546	.5543	.5540	.5537	.5534	.5531	.5528	
1	.5525	.5522	.5519	.5516	.5513	.5510	.5507	.5504	.5501	.5498	
2	.5495	.5491	.5488	.5485	.5482	.5479	.5476	.5473	.5470	.5467	
3	.5464	.5461	.5459	.5456	.5453	.5450	.5447	.5444	.5441	.5438	
4	.5435	.5432	.5429	.5426	.5423	.5420	.5417	.5414	.5411	.5408	
1.85	.5405	.5402	.5400	.5397	.5394	.5391	.5388	.5385	.5382	.5379	
6	.5376	.5373	.5371	.5368	.5365	.5362	.5359	.5356	.5353	.5350	
7	.5348	.5345	.5342	.5339	.5336	.5333	.5330	.5328	.5325	.5322	
8	.5319	.5316	.5313	.5311	.5308	.5305	.5302	.5299	.5297	.5294	
9	.5291	.5288	.5285	.5283	.5280	.5277	.5274	.5271	.5269	.5266	
1.90	.5263	.5260	.5258	.5255	.5252	.5249	.5247	.5244	.5241	.5238	
1	.5236	.5233	.5230	.5227	.5225	.5222	.5219	.5216	.5214	.5211	
2	.5208	.5206	.5203	.5200	.5198	.5195	.5192	.5189	.5187	.5184	
3	.5181	.5179	.5176	.5173	.5171	.5168	.5165	.5163	.5160	.5157	
4	.5155	.5152	.5149	.5147	.5144	.5141	.5139	.5136	.5133	.5131	
1.95	.5128	.5126	.5123	.5120	5118	.5115	.5112	.5110	.5107	.5105	
6	.5102	.5099	.5097	.5094	.5092	.5089	.5086	.5084	.5081	.5079	
7	.5076	.5074	.5071	.5068	.5066	.5063	.5061	.5058	.5056	.5053	−2
8	.5051	.5048	.5045	.5043	.5040	.5038	.5035	.5033	.5030	.5028	
9	.5025	.5023	.5020	.5018	.5015	.5013	.5010	.5008	.5005	.5003	

Moving the decimal point in either direction in *N* requires moving it in the OPPOSITE direction in body of table (see p. 26).

RECIPROCALS (*continued*)

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
2.0	.5000	.4975	.4950	.4926	.4902	.4878	.4854	.4831	.4808	.4785	– 24
1	.4762	.4739	.4717	.4695	.4673	.4651	.4630	.4608	.4587	.4566	– 21
2	.4545	.4525	.4505	.4484	.4464	.4444	.4425	.4405	.4386	.4367	– 20
3	.4348	.4329	.4310	.4292	.4274	.4255	.4237	.4219	.4202	.4184	– 18
4	.4167	.4149	.4132	.4115	.4098	.4082	.4065	.4049	.4032	.4016	– 17
2.5	.4000	.3984	.3968	.3953	.3937	.3922	.3906	.3891	.3876	.3861	– 15
6	.3846	.3831	.3817	.3802	.3788	.3774	.3759	.3745	.3731	.3717	– 14
7	.3704	.3690	.3676	.3663	.3650	.3636	.3623	.3610	.3597	.3584	– 13
8	.3571	.3559	.3546	.3534	.3521	.3509	.3497	.3484	.3472	.3460	– 12
9	.3448	.3436	.3425	.3413	.3401	.3390	.3378	.3367	.3356	.3344	– 12
3.0	.3333	.3322	.3311	.3300	.3289	.3279	.3268	.3257	.3247	.3236	– 11
1	.3226	.3215	.3205	.3195	.3185	.3175	.3165	.3155	.3145	.3135	– 10
2	.3125	.3115	.3106	.3096	.3086	.3077	.3067	.3058	.3049	.3040	– 10
3	.3030	.3021	.3012	.3003	.2994	.2985	.2976	.2967	.2959	.2950	– 9
4	.2941	.2933	.2924	.2915	.2907	.2899	.2890	.2882	.2874	.2865	– 8
3.5	.2857	.2849	.2841	.2833	.2825	.2817	.2809	.2801	.2793	.2786	– 8
6	.2778	.2770	.2762	.2755	.2747	.2740	.2732	.2725	.2717	.2710	– 8
7	.2703	.2695	.2688	.2681	.2674	.2667	.2660	.2653	.2646	.2639	– 7
8	.2632	.2625	.2618	.2611	.2604	.2597	.2591	.2584	.2577	.2571	– 7
9	.2564	.2558	.2551	.2545	.2538	.2532	.2525	.2519	.2513	.2506	– 6
4.0	.2500	.2494	.2488	.2481	.2475	.2469	.2463	.2457	.2451	.2445	– 6
1	.2439	.2433	.2427	.2421	.2415	.2410	.2404	.2398	.2392	.2387	– 6
2	.2381	.2375	.2370	.2364	.2358	.2353	.2347	.2342	.2336	.2331	– 6
3	.2326	.2320	.2315	.2309	.2304	.2299	.2294	.2288	.2283	.2278	– 5
4	.2273	.2268	.2262	.2257	.2252	.2247	.2242	.2237	.2232	.2227	– 5
4.5	.2222	.2217	.2212	.2208	.2203	.2198	.2193	.2188	.2183	.2179	– 5
6	.2174	.2169	.2165	.2160	.2155	.2151	.2146	.2141	.2137	.2132	– 5
7	.2128	.2123	.2119	.2114	.2110	.2105	.2101	.2096	.2092	.2088	– 4
8	.2083	.2079	.2075	.2070	.2066	.2062	.2058	.2053	.2049	.2045	– 4
9	.2041	.2037	.2033	.2028	.2024	.2020	.2016	.2012	.2008	.2004	– 4

$1/\pi = 0.318310 \qquad 1/e = 0.367879$

Explanation of Table of Reciprocals (pp. 24–27).

This table gives the values of $1/N$ for values of N from 1 to 10, correct to four figures. (Interpolated values may be in error by 1 in the fourth figure.)

To find the reciprocal of a number N outside the range from 1 to 10, note that moving the decimal point any number of places in either direction in column N is equivalent to moving it the same number of places in the **opposite direction** in the body of the table. For example:

$$\frac{1}{3.217} = 0.3108; \quad \frac{1}{3217.} = 0.000\,3108; \quad \frac{1}{0.003217} = 310.8$$

RECIPROCALS *(continued)*

N	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	.2000	.1996	.1992	.1988	.1984	.1980	.1976	.1972	.1969	.1965	− 4
.1	.1961	.1957	.1953	.1949	.1946	.1942	.1938	.1934	.1931	.1927	
.2	.1923	.1919	.1916	.1912	.1908	.1905	.1901	.1898	.1894	.1890	
.3	.1887	.1883	.1880	.1876	.1873	.1869	.1866	.1862	.1859	.1855	
.4	.1852	.1848	.1845	.1842	.1838	.1835	.1832	.1828	.1825	.1821	− 3
5.5	.1818	.1815	.1812	.1808	.1805	.1802	.1799	.1795	.1792	.1789	
.6	.1786	.1783	.1779	.1776	.1773	.1770	.1767	.1764	.1761	.1757	
.7	.1754	.1751	.1748	.1745	.1742	.1739	.1736	.1733	.1730	.1727	
.8	.1724	.1721	.1718	.1715	.1712	.1709	.1706	.1704	.1701	.1698	
.9	.1695	.1692	.1689	.1686	.1684	.1681	.1678	.1675	.1672	.1669	
6.0	.1667	.1664	.1661	.1658	.1656	.1653	.1650	.1647	.1645	.1642	
.1	.1639	.1637	.1634	.1631	.1629	.1626	.1623	.1621	.1618	.1616	
.2	.1613	.1610	.1608	.1605	.1603	.1600	.1597	.1595	.1592	.1590	
.3	.1587	.1585	.1582	.1580	.1577	.1575	.1572	.1570	.1567	.1565	− 2
.4	.1563	.1560	.1558	.1555	.1553	.1550	.1548	.1546	.1543	.1541	
6.5	.1538	.1536	.1534	.1531	.1529	.1527	.1524	.1522	.1520	.1517	
.6	.1515	.1513	.1511	.1508	.1506	.1504	.1502	.1499	.1497	.1495	
.7	.1493	.1490	.1488	.1486	.1484	.1481	.1479	.1477	.1475	.1473	
.8	.1471	.1468	.1466	.1464	.1462	.1460	.1458	.1456	.1453	.1451	
.9	.1449	.1447	.1445	.1443	.1441	.1439	.1437	.1435	.1433	.1431	
7.0	.1429	.1427	.1425	.1422	.1420	.1418	.1416	.1414	.1412	.1410	
.1	.1408	.1406	.1404	.1403	.1401	.1399	.1397	.1395	.1393	.1391	
.2	.1389	.1387	.1385	.1383	.1381	.1379	.1377	.1376	.1374	.1372	
.3	.1370	.1368	.1366	.1364	.1362	.1361	.1359	.1357	.1355	.1353	
.4	.1351	.1350	.1348	.1346	.1344	.1342	.1340	.1339	.1337	.1335	
7.5	.1333	.1332	.1330	.1328	.1326	.1325	.1323	.1321	.1319	.1318	
.6	.1316	.1314	.1312	.1311	.1309	.1307	.1305	.1304	.1302	.1300	
.7	.1299	.1297	.1295	.1294	.1292	.1290	.1289	.1287	.1285	.1284	
.8	.1282	.1280	.1279	.1277	.1276	.1274	.1272	.1271	.1269	.1267	
.9	.1266	.1264	.1263	.1261	.1259	.1258	.1256	.1255	.1253	.1252	
8.0	.1250	.1248	.1247	.1245	.1244	.1242	.1241	.1239	.1238	.1236	
.1	.1235	.1233	.1232	.1230	.1229	.1227	.1225	.1224	.1222	.1221	
.2	.1220	.1218	.1217	.1215	.1214	.1212	.1211	.1209	.1208	.1206	
.3	.1205	.1203	.1202	.1200	.1199	.1198	.1196	.1195	.1193	.1192	
.4	.1190	.1189	.1188	.1186	.1185	.1183	.1182	.1181	.1179	.1178	− 1
8.5	.1176	.1175	.1174	.1172	.1171	.1170	.1168	.1167	.1166	.1164	
.6	.1163	.1161	.1160	.1159	.1157	.1156	.1155	.1153	.1152	.1151	
.7	.1149	.1148	.1147	.1145	.1144	.1143	.1142	.1140	.1139	.1138	
.8	.1136	.1135	.1134	.1133	.1131	.1130	.1129	.1127	.1126	.1125	
.9	.1124	.1122	.1121	.1120	.1119	.1117	.1116	.1115	.1114	.1112	
9.0	.1111	.1110	.1109	.1107	.1106	.1105	.1104	.1103	.1101	.1100	
.1	.1099	.1098	.1096	.1095	.1094	.1093	.1092	.1091	.1089	.1088	
.2	.1087	.1086	.1085	.1083	.1082	.1081	.1080	.1079	.1078	.1076	
.3	.1075	.1074	.1073	.1072	.1071	.1070	.1068	.1067	.1066	.1065	
.4	.1064	.1063	.1062	.1060	.1059	.1058	.1057	.1056	.1055	.1054	
9.5	.1053	.1052	.1050	.1049	.1048	.1047	.1046	.1045	.1044	.1043	
.6	.1042	.1041	.1040	.1038	.1037	.1036	.1035	.1034	.1033	.1032	
.7	.1031	.1030	.1029	.1028	.1027	.1026	.1025	.1024	.1022	.1021	
.8	.1020	.1019	.1018	.1017	.1016	.1015	.1014	.1013	.1012	.1011	
.9	.1010	.1009	.1008	.1007	.1006	.1005	.1004	.1003	.1002	.1001	

Moving the decimal point in either direction in *N* requires moving it in the OPPOSITE direction in body of table (see p. 26).

CIRCUMFERENCES OF CIRCLES BY HUNDREDTHS

(For circumferences by eighths, see p. 32)

D	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	3.142	3.173	3.204	3.236	3.267	3.299	3.330	3.362	3.393	3.424	31
.1	3.456	3.487	3.519	3.550	3.581	3.613	3.644	3.676	3.707	3.738	
.2	3.770	3.801	3.833	3.864	3.896	3.927	3.958	3.990	4.021	4.053	
.3	4.084	4.115	4.147	4.178	4.210	4.241	4.273	4.304	4.335	4.367	
.4	4.398	4.430	4.461	4.492	4.524	4.555	4.587	4.618	4.650	4.681	
1.5	4.712	4.744	4.775	4.807	4.838	4.869	4.901	4.932	4.964	4.995	
.6	5.027	5.058	5.089	5.121	5.152	5.184	5.215	5.246	5.278	5.309	
.7	5.341	5.372	5.404	5.435	5.466	5.498	5.529	5.561	5.592	5.623	
.8	5.655	5.686	5.718	5.749	5.781	5.812	5.843	5.875	5.906	5.938	
.9	5.969	6.000	6.032	6.063	6.095	6.126	6.158	6.189	6.220	6.252	
2.0	6.283	6.315	6.346	6.377	6.409	6.440	6.472	6.503	6.535	6.566	
.1	6.597	6.629	6.660	6.692	6.723	6.754	6.786	6.817	6.849	6.880	
.2	6.912	6.943	6.974	7.006	7.037	7.069	7.100	7.131	7.163	7.194	
.3	7.226	7.257	7.288	7.320	7.351	7.383	7.414	7.446	7.477	7.508	
.4	7.540	7.571	7.603	7.634	7.665	7.697	7.728	7.760	7.791	7.823	
2.5	7.854	7.885	7.917	7.948	7.980	8.011	8.042	8.074	8.105	8.137	
.6	8.168	8.200	8.231	8.262	8.294	8.325	8.357	8.388	8.419	8.451	
.7	8.482	8.514	8.545	8.577	8.608	8.639	8.671	8.702	8.734	8.765	
.8	8.796	8.828	8.859	8.891	8.922	8.954	8.985	9.016	9.048	9.079	
.9	9.111	9.142	9.173	9.205	9.236	9.268	9.299	9.331	9.362	9.393	
3.0	9.425	9.456	9.488	9.519	9.550	9.582	9.613	9.645	9.676	9.708	
.1	9.739	9.770	9.802	9.833	9.865	9.896	9.927	9.959	9.990	10.022	31
1										10.02	3
.2	10.05	10.08	10.12	10.15	10.18	10.21	10.24	10.27	10.30	10.34	
.3	10.37	10.40	10.43	10.46	10.49	10.52	10.56	10.59	10.62	10.65	
.4	10.68	10.71	10.74	10.78	10.81	10.84	10.87	10.90	10.93	10.96	
3.5	11.00	11.03	11.06	11.09	11.12	11.15	11.18	11.22	11.25	11.28	
.6	11.31	11.34	11.37	11.40	11.44	11.47	11.50	11.53	11.56	11.59	
.7	11.62	11.66	11.69	11.72	11.75	11.78	11.81	11.84	11.88	11.91	
.8	11.94	11.97	12.00	12.03	12.06	12.10	12.13	12.16	12.19	12.22	
.9	12.25	12.28	12.32	12.35	12.38	12.41	12.44	12.47	12.50	12.53	
4.0	12.57	12.60	12.63	12.66	12.69	12.72	12.75	12.79	12.82	12.85	
.1	12.88	12.91	12.94	12.97	13.01	13.04	13.07	13.10	13.13	13.16	
.2	13.19	13.23	13.26	13.29	13.32	13.35	13.38	13.41	13.45	13.48	
.3	13.51	13.54	13.57	13.60	13.63	13.67	13.70	13.73	13.76	13.79	
.4	13.82	13.85	13.89	13.92	13.95	13.98	14.01	14.04	14.07	14.11	
4.5	14.14	14.17	14.20	14.23	14.26	14.29	14.33	14.36	14.39	14.42	
.6	14.45	14.48	14.51	14.55	14.58	14.61	14.64	14.67	14.70	14.73	
.7	14.77	14.80	14.83	14.86	14.89	14.92	14.95	14.99	15.02	15.05	
.8	15.08	15.11	15.14	15.17	15.21	15.24	15.27	15.30	15.33	15.36	
.9	15.39	15.43	15.46	15.49	15.52	15.55	15.58	15.61	15.65	15.68	

Explanation of Table of Circumferences (pp. 28–29)

This table gives the product of π times any number D from 1 to 10; that is, it is a table of multiples of π. (D = diameter.)

Moving the decimal point one place in column D is equivalent to moving it one place in the body of the table.

$$\text{Circumference} = \pi \times \text{diam.} = 3.141593 \times \text{diam.}$$

Conversely,

$$\text{Diameter} = \frac{1}{\pi} \times \text{circumf.} = 0.31831 \times \text{circumf.}$$

CIRCUMFERENCES BY HUNDREDTHS *(continued)*

D	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	15.71	15.74	15.77	15.80	15.83	15.87	15.90	15.93	15.96	15.99	3
.1	16.02	16.05	16.08	16.12	16.15	16.18	16.21	16.24	16.27	16.30	
.2	16.34	16.37	16.40	16.43	16.46	16.49	16.52	16.56	16.59	16.62	
.3	16.65	16.68	16.71	16.74	16.78	16.81	16.84	16.87	16.90	16.93	
.4	16.96	17.00	17.03	17.06	17.09	17.12	17.15	17.18	17.22	17.25	
5.5	17.28	17.31	17.34	17.37	17.40	17.44	17.47	17.50	17.53	17.56	
.6	17.59	17.62	17.66	17.69	17.72	17.75	17.78	17.81	17.84	17.88	
.7	17.91	17.94	17.97	18.00	18.03	18.06	18.10	18.13	18.16	18.19	
.8	18.22	18.25	18.28	18.32	18.35	18.38	18.41	18.44	18.47	18.50	
.9	18.54	18.57	18.60	18.63	18.66	18.69	18.72	18.76	18.79	18.82	
6.0	18.85	18.88	18.91	18.94	18.98	19.01	19.04	19.07	19.10	19.13	
.1	19.16	19.20	19.23	19.26	19.29	19.32	19.35	19.38	19.42	19.45	
.2	19.48	19.51	19.54	19.57	19.60	19.63	19.67	19.70	19.73	19.76	
.3	19.79	19.82	19.85	19.89	19.92	19.95	19.98	20.01	20.04	20.07	
.4	20.11	20.14	20.17	20.20	20.23	20.26	20.29	20.33	20.36	20.39	
6.5	20.42	20.45	20.48	20.51	20.55	20.58	20.61	20.64	20.67	20.70	
.6	20.73	20.77	20.80	20.83	20.86	20.89	20.92	20.95	20.99	21.02	
.7	21.05	21.08	21.11	21.14	21.17	21.21	21.24	21.27	21.30	21.33	
.8	21.36	21.39	21.43	21.46	21.49	21.52	21.55	21.58	21.61	21.65	
.9	21.68	21.71	21.74	21.77	21.80	21.83	21.87	21.90	21.93	21.96	
7.0	21.99	22.02	22.05	22.09	22.12	22.15	22.18	22.21	22.24	22.27	
.1	22.31	22.34	22.37	22.40	22.43	22.46	22.49	22.53	22.56	22.59	
.2	22.62	22.65	22.68	22.71	22.75	22.78	22.81	22.84	22.87	22.90	
.3	22.93	22.97	23.00	23.03	23.06	23.09	23.12	23.15	23.18	23.22	
.4	23.25	23.28	23.31	23.34	23.37	23.40	23.44	23.47	23.50	23.53	
7.5	23.56	23.59	23.62	23.66	23.69	23.72	23.75	23.78	23.81	23.84	
.6	23.88	23.91	23.94	23.97	24.00	24.03	24.06	24.10	24.13	24.16	
.7	24.19	24.22	24.25	24.28	24.32	24.35	24.38	24.41	24.44	24.47	
.8	24.50	24.54	24.57	24.60	24.63	24.66	24.69	24.72	24.76	24.79	
.9	24.82	24.85	24.88	24.91	24.94	24.98	25.01	25.04	25.07	25.10	
8.0	25.13	25.16	25.20	25.23	25.26	25.29	25.32	25.35	25.38	25.42	
.1	25.45	25.48	25.51	25.54	25.57	25.60	25.64	25.67	25.70	25.73	
.2	25.76	25.79	25.82	25.86	25.89	25.92	25.95	25.98	26.01	26.04	
.3	26.08	26.11	26.14	26.17	26.20	26.23	26.26	26.30	26.33	26.36	
.4	26.39	26.42	26.45	26.48	26.52	26.55	26.58	26.61	26.64	26.67	
8.5	26.70	26.73	26.77	26.80	26.83	26.86	26.89	26.92	26.95	26.99	
.6	27.02	27.05	27.08	27.11	27.14	27.17	27.21	27.24	27.27	27.30	
7	27.33	27.36	27.39	27.43	27.46	27.49	27.52	27.55	27.58	27.61	
.8	27.65	27.68	27.71	27.74	27.77	27.80	27.83	27.87	27.90	27.93	
.9	27.96	27.99	28.02	28.05	28.09	28.12	28.15	28.18	28.21	28.24	
9.0	28.27	28.31	28.34	28.37	28.40	28.43	28.46	28.49	28.53	28.56	
.1	28.59	28.62	28.65	28.68	28.71	28.75	28.78	28.81	28.84	28.87	
.2	28.90	28.93	28.97	29.00	29.03	29.06	29.09	29.12	29.15	29.19	
.3	29.22	29.25	29.28	29.31	29.34	29.37	29.41	29.44	29.47	29.50	
.4	29.53	29.56	29.59	29.63	29.66	29.69	29.72	29.75	29.78	29.81	
9.5	29.85	29.88	29.91	29.94	29.97	30.00	30.03	30.07	30.10	30.13	
.6	30.16	30.19	30.22	30.25	30.28	30.32	30.35	30.38	30.41	30.44	
.7	30.47	30.50	30.54	30.57	30.60	30.63	30.66	30.69	30.72	30.76	
.8	30.79	30.82	30.85	30.88	30.91	30.94	30.98	31.01	31.04	31.07	
.9	31.10	31.13	31.16	31.20	31.23	31.26	31.29	31.32	31.35	31.38	
10.0	31.42										

Moving the decimal point ONE place in *D* requires moving it ONE place in body of table (see p. 28).

AREAS OF CIRCLES BY HUNDREDTHS

(For areas by eighths, see p. 32)

D	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	0.785	0.801	0.817	0.833	0.849	0.866	0.882	0.899	0.916	0.933	16
.1	0.950	0.968	0.985	1.003	1.021	1.039	1.057	1.075	1.094	1.112	18
.2	1.131	1.150	1.169	1.188	1.208	1.227	1.247	1.267	1.287	1.307	20
.3	1.327	1.348	1.368	1.389	1.410	1.431	1.453	1.474	1.496	1.517	21
.4	1.539	1.561	1.584	1.606	1.629	1.651	1.674	1.697	1.720	1.744	23
1.5	1.767	1.791	1.815	1.839	1.863	1.887	1.911	1.936	1.961	1.986	24
.6	2.011	2.036	2.061	2.087	2.112	2.138	2.164	2.190	2.217	2.243	26
.7	2.270	2.297	2.324	2.351	2.378	2.405	2.433	2.461	2.488	2.516	27
.8	2.545	2.573	2.602	2.630	2.659	2.688	2.717	2.746	2.776	2.806	29
.9	2.835	2.865	2.895	2.926	2.956	2.986	3.017	3.048	3.079	3.110	31
2.0	3.142	3.173	3.205	3.237	3.269	3.301	3.333	3.365	3.398	3.431	32
.1	3.464	3.497	3.530	3.563	3.597	3.631	3.664	3.698	3.733	3.767	34
.2	3.801	3.836	3.871	3.906	3.941	3.976	4.011	4.047	4.083	4.119	35
.3	4.155	4.191	4.227	4.264	4.301	4.337	4.374	4.412	4.449	4.486	37
.4	4.524	4.562	4.600	4.638	4.676	4.714	4.753	4.792	4.831	4.870	38
2.5	4.909	4.948	4.988	5.027	5.067	5.107	5.147	5.187	5.228	5.269	40
.6	5.309	5.350	5.391	5.433	5.474	5.515	5.557	5.599	5.641	5.683	42
.7	5.726	5.768	5.811	5.853	5.896	5.940	5.983	6.026	6.070	6.114	43
.8	6.158	6.202	6.246	6.290	6.335	6.379	6.424	6.469	6.514	6.560	45
.9	6.605	6.651	6.697	6.743	6.789	6.835	6.881	6.928	6.975	7.022	46
3.0	7.069	7.116	7.163	7.211	7.258	7.306	7.354	7.402	7.451	7.499	48
.1	7.548	7.596	7.645	7.694	7.744	7.793	7.843	7.892	7.942	7.992	49
.2	8.042	8.093	8.143	8.194	8.245	8.296	8.347	8.398	8.450	8.501	51
.3	8.553	8.605	8.657	8.709	8.762	8.814	8.867	8.920	8.973	9.026	53
.4	9.079	9.133	9.186	9.240	9.294	9.348	9.402	9.457	9.511	9.566	54
3.5	9.621	9.676	9.731	9.787	9.842	9.898	9.954	10.010			56
.5								10.01	10.07	10.12	6
.6	10.18	10.24	10.29	10.35	10.41	10.46	10.52	10.58	10.64	10.69	6
.7	10.75	10.81	10.87	10.93	10.99	11.04	11.10	11.16	11.22	11.28	
.8	11.34	11.40	11.46	11.52	11.58	11.64	11.70	11.76	11.82	11.88	
.9	11.95	12.01	12.07	12.13	12.19	12.25	12.32	12.38	12.44	12.50	
4.0	12.57	12.63	12.69	12.76	12.82	12.88	12.95	13.01	13.07	13.14	7
.1	13.20	13.27	13.33	13.40	13.46	13.53	13.59	13.66	13.72	13.79	
.2	13.85	13.92	13.99	14.05	14.12	14.19	14.25	14.32	14.39	14.45	
.3	14.52	14.59	14.66	14.73	14.79	14.86	14.93	15.00	15.07	15.14	
.4	15.21	15.27	15.34	15.41	15.48	15.55	15.62	15.69	15.76	15.83	
4.5	15.90	15.98	16.05	16.12	16.19	16.26	16.33	16.40	16.47	16.55	
.6	16.62	16.69	16.76	16.84	16.91	16.98	17.06	17.13	17.20	17.28	
.7	17.35	17.42	17.50	17.57	17.65	17.72	17.80	17.87	17.95	18.02	
.8	18.10	18.17	18.25	18.32	18.40	18.47	18.55	18.63	18.70	18.78	8
.9	18.86	18.93	19.01	19.09	19.17	19.24	19.32	19.40	19.48	19.56	

Explanation of Table of Areas of Circles (pp. 30–31)

Moving the decimal point **one** place in column *D* is equivalent to moving it **two** places in the body of the table. (*D* = diameter.)

$$\text{Area of circle} = \frac{\pi}{4} \times (\text{diam.}^2) = 0.785398 \times (\text{diam.}^2)$$

Conversely,

$$\text{Diam.} = \sqrt{\frac{4}{\pi}} \times \sqrt{\text{area}} = 1.128379 \times \sqrt{\text{area}}$$

AREAS OF CIRCLES BY HUNDREDTHS (*continued*)

D	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	19.63	19.71	19.79	19.87	19.95	20.03	20.11	20.19	20.27	20.35	8
.1	20.43	20.51	20.59	20.67	20.75	20.83	20.91	20.99	21.07	21.16	
.2	21.24	21.32	21.40	21.48	21.57	21.65	21.73	21.81	21.90	21.98	
.3	22.06	22.15	22.23	22.31	22.40	22.48	22.56	22.65	22.73	22.82	
.4	22.90	22.99	23.07	23.16	23.24	23.33	23.41	23.50	23.59	23.67	9
5.5	23.76	23.84	23.93	24.02	24.11	24.19	24.28	24.37	24.45	24.54	
.6	24.63	24.72	24.81	24.89	24.98	25.07	25.16	25.25	25.34	25.43	
.7	25.52	25.61	25.70	25.79	25.88	25.97	26.06	26.15	26.24	26.33	
.8	26.42	26.51	26.60	26.69	26.79	26.88	26.97	27.06	27.15	27.25	
.9	27.34	27.43	27.53	27.62	27.71	27.81	27.90	27.99	28.09	28.18	
6.0	28.27	28.37	28.46	28.56	28.65	28.75	28.84	28.94	29.03	29.13	10
.1	29.22	29.32	29.42	29.51	29.61	29.71	29.80	29.90	30.00	30.09	
.2	30.19	30.29	30.39	30.48	30.58	30.68	30.78	30.88	30.97	31.07	
.3	31.17	31.27	31.37	31.47	31.57	31.67	31.77	31.87	31.97	32.07	
.4	32.17	32.27	32.37	32.47	32.57	32.67	32.78	32.88	32.98	33.08	
6.5	33.18	33.29	33.39	33.49	33.59	33.70	33.80	33.90	34.00	34.11	
.6	34.21	34.32	34.42	34.52	34.63	34.73	34.84	34.94	35.05	35.15	
.7	35.26	35.36	35.47	35.57	35.68	35.78	35.89	36.00	36.10	36.21	11
.8	36.32	36.42	36.53	36.64	36.75	36.85	36.96	37.07	37.18	37.28	
.9	37.39	37.50	37.61	37.72	37.83	37.94	38.05	38.16	38.26	38.37	
7.0	38.48	38.59	38.70	38.82	38.93	39.04	39.15	39.26	39.37	39.48	
.1	39.59	39.70	39.82	39.93	40.04	40.15	40.26	40.38	40.49	40.60	
.2	40.72	40.83	40.94	41.06	41.17	41.28	41.40	41.51	41.62	41.74	
.3	41.85	41.97	42.08	42.20	42.31	42.43	42.54	42.66	42.78	42.89	12
.4	43.01	43.12	43.24	43.36	43.47	43.59	43.71	43.83	43.94	44.06	
7.5	44.18	44.30	44.41	44.53	44.65	44.77	44.89	45.01	45.13	45.25	
.6	45.36	45.48	45.60	45.72	45.84	45.96	46.08	46.20	46.32	46.45	
.7	46.57	46.69	46.81	46.93	47.05	47.17	47.29	47.42	47.54	47.66	
.8	47.78	47.91	48.03	48.15	48.27	48.40	48.52	48.65	48.77	48.89	
.9	49.02	49.14	49.27	49.39	49.51	49.64	49.76	49.89	50.01	50.14	
8.0	50.27	50.39	50.52	50.64	50.77	50.90	51.02	51.15	51.28	51.40	13
.1	51.53	51.66	51.78	51.91	52.04	52.17	52.30	52.42	52.55	52.68	
.2	52.81	52.94	53.07	53.20	53.33	53.46	53.59	53.72	53.85	53.98	
.3	54.11	54.24	54.37	54.50	54.63	54.76	54.89	55.02	55.15	55.29	
.4	55.42	55.55	55.68	55.81	55.95	56.08	56.21	56.35	56.48	56.61	
8.5	56.75	56.88	57.01	57.15	57.28	57.41	57.55	57.68	57.82	57.95	
.6	58.09	58.22	58.36	58.49	58.63	58.77	58.90	59.04	59.17	59.31	14
.7	59.45	59.58	59.72	59.86	59.99	60.13	60.27	60.41	60.55	60.68	
.8	60.82	60.96	61.10	61.24	61.38	61.51	61.65	61.79	61.93	62.07	
.9	62.21	62.35	62.49	62.63	62.77	62.91	63.05	63.19	63.33	63.48	
9.0	63.62	63.76	63.90	64.04	64.18	64.33	64.47	64.61	64.75	64.90	
.1	65.04	65.18	65.33	65.47	65.61	65.76	65.90	66.04	66.19	66.33	15
.2	66.48	66.62	66.77	66.91	67.06	67.20	67.35	67.49	67.64	67.78	
.3	67.93	68.08	68.22	68.37	68.51	68.66	68.81	68.96	69.10	69.25	
.4	69.40	69.55	69.69	69.84	69.99	70.14	70.29	70.44	70.58	70.73	
9.5	70.88	71.03	71.18	71.33	71.48	71.63	71.78	71.93	72.08	72.23	
.6	72.38	72.53	72.68	72.84	72.99	73.14	73.29	73.44	73.59	73.75	
.7	73.90	74.05	74.20	74.36	74.51	74.66	74.82	74.97	75.12	75.28	
.8	75.43	75.58	75.74	75.89	76.05	76.20	76.36	76.51	76.67	76.82	
.9	76.98	77.13	77.29	77.44	77.60	77.76	77.91	78.07	78.23	78.38	16

Moving the decimal point ONE place in *D* requires moving it TWO places in body of table (see p. 30).

CIRCUMFERENCES AND AREAS OF CIRCLES BY EIGHTHS, ETC.

(For tenths, see p. 28)

Diam.	Circum.	Area	Diam.	Circum.	Area	Diam.	Circum.	Area	Diam.	Circum.	Area
			7/8	2.749	.6013	4	12.57	12.57	9	28.27	63.62
1/64	.04909	.00019	57/64	2.798	.6230	1/16	12.76	12.96	1/8	28.67	65.40
1/32	.09817	.00077	29/32	2.847	.6450	1/8	12.96	13.36	1/4	29.06	67.20
3/64	.1473	.00173	59/64	2.896	.6675	3/16	13.16	13.77	3/8	29.45	69.03
1/16	.1963	.00307	15/16	2.945	.6903	1/4	13.35	14.19	1/2	29.85	70.88
5/64	.2454	.00479	61/64	2.994	.7135	5/16	13.55	14.61	5/8	30.24	72.76
3/32	.2945	.00690	31/32	3.043	.7371	3/8	13.74	15.03	3/4	30.63	74.66
7/64	.3436	.00940	63/64	3.093	.7610	7/16	13.94	15.47	7/8	31.02	76.59
1/8	.3927	.01227	1	3.142	.7854	1/2	14.14	15.90	10	31.42	78.54
9/64	.4418	.01553	1/16	3.338	.8866	9/16	14.33	16.35	1/8	31.81	80.52
5/32	.4909	.01917	1/8	3.534	.9940	5/8	14.53	16.80	1/4	32.20	82.52
11/64	.5400	.02320	3/16	3.731	1.108	11/16	14.73	17.26	3/8	32.59	84.54
3/16	.5890	.02761	1/4	3.927	1.227	3/4	14.92	17.72	1/2	32.99	86.59
13/64	.6381	.03241	5/16	4.123	1.353	13/16	15.12	18.19	5/8	33.38	88.66
7/32	.6872	.03758	3/8	4.320	1.485	7/8	15.32	18.67	3/4	33.77	90.76
15/64	.7363	.04314	7/16	4.516	1.623	15/16	15.51	19.15	7/8	34.16	92.89
1/4	.7854	.04909	1/2	4.712	1.767	5	15.71	19.63	11	34.56	95.03
17/64	.8345	.05542	9/16	4.909	1.917	1/16	15.90	20.13	1/8	34.95	97.21
9/32	.8836	.06213	5/8	5.105	2.074	1/8	16.10	20.63	1/4	35.34	99.40
19/64	.9327	.06922	11/16	5.301	2.237	3/16	16.30	21.14	3/8	35.74	101.6
5/16	.9817	.07670	3/4	5.498	2.405	1/4	16.49	21.65	1/2	36.13	103.9
21/64	1.031	.08456	13/16	5.694	2.580	5/16	16.69	22.17	5/8	36.52	106.1
11/32	1.080	.09281	7/8	5.890	2.761	3/8	16.89	22.69	3/4	36.91	108.4
23/64	1.129	.1014	15/16	6.087	2.948	7/16	17.08	23.22	7/8	37.31	110.8
3/8	1.178	.1104	2	6.283	3.142	1/2	17.28	23.76	12	37.70	113.1
25/64	1.227	.1198	1/16	6.480	3.341	9/16	17.48	24.30	1/8	38.09	115.5
13/32	1.276	.1296	1/8	6.676	3.547	5/8	17.67	24.85	1/4	38.48	117.9
27/64	1.325	.1398	3/16	6.872	3.758	11/16	17.87	25.41	3/8	38.88	120.3
7/16	1.374	.1503	1/4	7.069	3.976	3/4	18.06	25.97	1/2	39.27	122.7
29/64	1.424	.1613	5/16	7.265	4.200	13/16	18.26	26.53	5/8	39.66	125.2
15/32	1.473	.1726	3/8	7.461	4.430	7/8	18.46	27.11	3/4	40.06	127.7
31/64	1.522	.1843	7/16	7.658	4.666	15/16	18.65	27.69	7/8	40.45	130.2
1/2	1.571	.1963	1/2	7.854	4.909	6	18.85	28.27	13	40.84	132.7
33/64	1.620	.2088	9/16	8.050	5.157	1/8	19.24	29.46	1/8	41.23	135.3
17/32	1.669	.2217	5/8	8.247	5.412	1/4	19.63	30.68	1/4	41.63	137.9
35/64	1.718	.2349	11/16	8.443	5.673	3/8	20.03	31.92	3/8	42.02	140.5
9/16	1.767	.2485	3/4	8.639	5.940	1/2	20.42	33.18	1/2	42.41	143.1
37/64	1.816	.2625	13/16	8.836	6.213	5/8	20.81	34.47	5/8	42.80	145.8
19/32	1.865	.2769	7/8	9.032	6.492	3/4	21.21	35.78	3/4	43.20	148.5
39/64	1.914	.2916	15/16	9.228	6.777	7/8	21.60	37.12	7/8	43.59	151.2
5/8	1.963	.3068	3	9.425	7.069	7	21.99	38.48	14	43.98	153.9
41/64	2.013	.3223	1/16	9.621	7.366	1/8	22.38	39.87	1/8	44.37	156.7
21/32	2.062	.3382	1/8	9.817	7.670	1/4	22.78	41.28	1/4	44.77	159.5
43/64	2.111	.3545	3/16	10.01	7.980	3/8	23.17	42.72	3/8	45.16	162.3
11/16	2.160	.3712	1/4	10.21	8.296	1/2	23.56	44.18	1/2	45.55	165.1
45/64	2.209	.3883	5/16	10.41	8.618	5/8	23.95	45.66	5/8	45.95	168.0
23/32	2.258	.4057	3/8	10.60	8.946	3/4	24.35	47.17	3/4	46.34	170.9
47/64	2.307	.4236	7/16	10.80	9.281	7/8	24.74	48.71	7/8	46.73	173.8
3/4	2.356	.4418	1/2	11.00	9.621	8	25.13	50.27	15	47.12	176.7
49/64	2.405	.4604	9/16	11.19	9.968	1/8	25.53	51.85	1/8	47.52	179.7
25/32	2.454	.4794	5/8	11.39	10.32	1/4	25.92	53.46	1/4	47.91	182.7
51/64	2.503	.4987	11/16	11.58	10.68	3/8	26.31	55.09	3/8	48.30	185.7
13/16	2.553	.5185	3/4	11.78	11.04	1/2	26.70	56.75	1/2	48.69	188.7
53/64	2.602	.5386	13/16	11.98	11.42	5/8	27.10	58.43	5/8	49.09	191.7
27/32	2.651	.5591	7/8	12.17	11.79	3/4	27.49	60.13	3/4	49.48	194.8
55/64	2.700	.5800	15/16	12.37	12.18	7/8	27.88	61.86	7/8	49.87	197.9

CIRCUMFERENCES AND AREAS BY EIGHTHS—(*continued*)

Diam.	Circum.	Area	Diam.	Circum.	Area	Diam.	Circum.	Area	Diam.	Circum.	Area
16	50.27	201.1	**19** ½	61.26	298.6	**23**	72.26	415.5	**29**	91.11	660.5
⅛	50.66	204.2	⅝	61.65	302.5	⅛	72.65	420.0	¼	91.89	672.0
¼	51.05	207.4	¾	62.05	306.4	¼	73.04	424.6	½	92.68	683.5
⅜	51.44	210.6	⅞	62.44	310.2	⅜	73.43	429.1	¾	93.46	695.1
½	51.84	213.8	**20**	62.83	314.2	½	73.83	433.7	**30**	94.25	706.9
⅝	52.23	217.1	⅛	63.22	318.1	⅝	74.22	438.4	¼	95.03	718.7
¾	52.62	220.4	¼	63.62	322.1	¾	74.61	443.0	½	95.82	730.6
⅞	53.01	223.7	⅜	64.01	326.1	⅞	75.01	447.7	¾	96.60	742.6
17	53.41	227.0	½	64.40	330.1	**24**	75.40	452.4	**31**	97.39	754.8
⅛	53.80	230.3	⅝	64.80	334.1	¼	76.18	461.9	¼	98.17	767.0
¼	54.19	233.7	¾	65.19	338.2	½	76.97	471.4	½	98.96	779.3
⅜	54.59	237.1	⅞	65.58	342.2	¾	77.75	481.1	¾	99.75	791.7
½	54.98	240.5	**21**	65.97	346.4	**25**	78.54	490.9	**32**	100.5	804.2
⅝	55.37	244.0	⅛	66.37	350.5	¼	79.33	500.7	¼	101.3	816.9
¾	55.76	247.4	¼	66.76	354.7	½	80.11	510.7	½	102.1	829.6
⅞	56.16	250.9	⅜	67.15	358.8	¾	80.90	520.8	¾	102.9	842.4
18	56.55	254.5	½	67.54	363.1	**26**	81.68	530.9	**33**	103.7	855.3
⅛	56.94	258.0	⅝	67.94	367.3	¼	82.47	541.2	¼	104.5	868.3
¼	57.33	261.6	¾	68.33	371.5	½	83.25	551.5	½	105.2	881.4
⅜	57.73	265.2	⅞	68.72	375.8	¾	84.04	562.0	¾	106.0	894.6
½	58.12	268.8	**22**	69.12	380.1	**27**	84.82	572.6	**34**	106.8	907.9
⅝	58.51	272.4	⅛	69.51	384.5	¼	85.61	583.2	¼	107.6	921.3
¾	58.90	276.1	¼	69.90	388.8	½	86.39	594.0	½	108.4	934.8
⅞	59.30	279.8	⅜	70.29	393.2	¾	87.18	604.8	¾	109.2	948.4
19	59.69	283.5	½	70.69	397.6	**28**	87.96	615.8	**35**	110.0	962.1
⅛	60.08	287.3	⅝	71.08	402.0	¼	88.75	626.8	¼	110.7	975.9
¼	60.48	291.0	¾	71.47	406.5	½	89.54	637.9	½	111.5	989.8
⅜	60.87	294.8	⅞	71.86	411.0	¾	90.32	649.2	¾	112.3	1003.8

AREAS OF CIRCLES. Diameters in Feet and Inches, Areas in Square Feet

Feet	Inches											
	0	1	2	3	4	5	6	7	8	9	10	11
0	.0000	.0055	.0218	.0491	.0873	.1364	.1963	.2673	.3491	.4418	.5454	.6600
1	.7854	.9218	1.069	1.227	1.396	1.576	1.767	1.969	2.182	2.405	2.640	2.885
2	3.142	3.409	3.687	3.976	4.276	4.587	4.909	5.241	5.585	5.940	6.305	6.681
3	7.069	7.467	7.876	8.296	8.727	9.168	9.621	10.08	10.56	11.04	11.54	12.05
4	12.57	13.10	13.64	14.19	14.75	15.32	15.90	16.50	17.10	17.72	18.35	18.99
5	19.63	20.29	20.97	21.65	22.34	23.04	23.76	24,48	25.22	25.97	26.73	27.49
6	28.27	29.07	29.87	30.68	31.50	32.34	33.18	34.04	34.91	35.78	36.67	37.57
7	38.48	39.41	40.34	41.28	42.24	43.20	44.18	45.17	46.16	47.17	48.19	49.22
8	50.27	51.32	52.38	53.46	54.54	55.64	56.75	57.86	58.99	60.13	61.28	62.44
9	63.62	64.80	66.00	67.20	68.42	69.64	70.88	72.13	73.39	74.66	75.94	77.24
10	78.54	79.85	81.18	82.52	83.86	85.22	86.59	87.97	89.36	90.76	92.18	93.60
11	95.03	96.48	97.93	99.40	100.9	102.4	103.9	105.4	106.9	108.4	110.0	111.5
12	113.1	114.7	116.3	117.9	119.5	121.1	122.7	124.4	126.0	127.7	129.4	131.0
13	132.7	134.4	136.2	137.9	139.6	141.4	143.1	144.9	146.7	148.5	150.3	152.1
14	153.9	155.8	157.6	159.5	161.4	163.2	165.1	167.0	168.9	170.9	172.8	174.8

If given diameter is not found in this table, reduce diameter to feet and decimals of a foot by aid of the following auxiliary table, and then find area from pp. 30–31.

From Inches and Fractions of an Inch to Decimals of a Foot

Inches	1	2	3	4	5	6	7	8	9	10	11
Feet	.0833	.1667	.2500	.3333	.4167	.5000	.5833	.6667	.7500	.8333	.9167

Inches	⅛	¼	⅜	½	⅝	¾	⅞
Feet	.0104	.0208	.0313	.0417	.0521	.0625	.0729

Example. 5 ft. 7⅜ in. = 5.0 + 0.5833 + 0.0313 = 5.6146 ft.

SEGMENTS OF CIRCLES, GIVEN h/c

Given: h = height; c = chord. (For explanation of this table, see p. 38)

$\frac{h}{c}$	$\frac{\text{Diam.}}{c}$	Diff.	$\frac{\text{Arc}}{c}$	Diff.	$\frac{\text{Area}}{h \times c}$	Diff.	Central angle, v	Diff.	$\frac{h}{\text{Diam.}}$	Diff.
.00			1.000	0	.6667	0	0.00°	458	.0000	4
1	25.010	12490	1.000	1	.6667	2	4.58	458	.0004	12
2	12.520	*4157	1.001	1	.6669	2	9.16	457	.0016	20
3	8.363	*2073	1.002	2	.6671	4	13.73	457	.0036	28
4	6.290	*1240	1.004	3	.6675	5	18.30	454	.0064	35
.05	5.050	*823	1.007	3	.6680	6	22.84°	453	.0099	43
6	4.227	*586	1.010	3	.6686	7	27.37	451	.0142	50
7	3.641	*436	1.013	4	.6693	8	31.88	448	.0192	58
8	3.205	*337	1.017	4	.6701	9	36.36	446	.0250	64
9	2.868	*268	1.021	5	.6710	10	40.82	442	.0314	71
.10	2.600	*217	1.026	6	.6720	11	45.24°	439	.0385	77
1	2.383	*180	1.032	6	.6731	12	49.63	435	.0462	83
2	2.203	*150	1.038	6	.6743	13	53.98	432	.0545	88
3	2.053	*127	1.044	7	.6756	14	58.30	427	.0633	94
4	1.926	*109	1.051	8	.6770	15	62.57	423	.0727	99
.15	1.817	*94	1.059	8	.6785	16	66.80°	418	.0826	103
6	1.723	*82	1.067	8	.6801	17	70.98	413	.0929	107
7	1.641	*72	1.075	9	.6818	18	75.11	409	.1036	111
8	1.569	*63	1.084	10	.6836	19	79.20	403	.1147	115
9	1.506	56	1.094	9	.6855	20	83.23	398	.1262	117
.20	1.450	50	1.103	11	.6875	21	87.21°	392	.1379	120
1	1.400	44	1.114	10	.6896	22	91.13	387	.1499	123
2	1.356	39	1.124	12	.6918	23	95.00	381	.1622	124
3	1.317	35	1.136	11	.6941	24	98.81	375	.1746	127
4	1.282	32	1.147	12	.6965	24	102.56	370	.1873	127
.25	1.250	28	1.159	12	.6989	25	106.26°	364	.2000	128
6	1.222	26	1.171	13	.7014	27	109.90	358	.2128	130
7	1.196	23	1.184	13	.7041	27	113.48	352	.2258	129
8	1.173	21	1.197	14	.7068	28	117.00	345	.2387	130
9	1.152	19	1.211	14	.7096	29	120.45	341	.2517	130
.30	1.133	17	1.225	14	.7125	29	123.86°	334	.2647	130
1	1.116	15	1.239	15	.7154	31	127.20	328	.2777	129
2	1.101	13	1.254	15	.7185	31	130.48	322	.2906	128
3	1.088	13	1.269	15	.7216	32	133.70	316	.3034	128
4	1.075	11	1.284	16	.7248	32	136.86	311	.3162	127
.35	1.064	10	1.300	16	.7280	34	139.97°	305	.3289	125
6	1.054	8	1.316	16	.7314	34	143.02	299	.3414	124
7	1.046	8	1.332	17	.7348	35	146.01	293	.3538	123
8	1.038	7	1.349	17	.7383	36	148.94	288	.3661	122
9	1.031	6	1.366	17	.7419	36	151.82	282	.3783	119
.40	1.025	5	1.383	18	.7455	37	154.64°	277	.3902	119
1	1.020	5	1.401	18	.7492	38	157.41	271	.4021	116
2	1.015	4	1.419	18	.7530	38	160.12	266	.4137	115
3	1.011	3	1.437	18	.7568	39	162.78	261	.4252	112
4	1.008	2	1.455	19	.7607	40	165.39	256	.4364	111
.45	1.006	3	1.474	19	.7647	40	167.95°	251	.4475	109
6	1.003	1	1.493	19	.7687	41	170.46	245	.4584	107
7	1.002	1	1.512	19	.7728	41	172.91	241	.4691	105
8	1.001	1	1.531	20	.7769	42	175.32	237	.4796	103
9	1.000	0	1.551	20	.7811	43	177.69	231	.4899	101
.50	1.000		1.571		.7854		180.00°		.5000	

* Interpolation may be inaccurate at these points.

SEGMENTS OF CIRCLES, GIVEN h/D

Given: h = height; D = diameter of circle. (For explanation of this table, see p. 38)

$\frac{h}{D}$	$\frac{\text{Arc}}{D}$	Diff.	$\frac{\text{Area}}{D^2}$	Diff.	Central angle, v	Diff.	$\frac{\text{Chord}}{D}$	Diff.	$\frac{\text{Arc}}{\text{Circumf.}}$	Diff.	$\frac{\text{Area}}{\text{Circle}}$	Diff.
.00	0.0000	2003	.0000	13	0.00°	2296	.0000	*1990	.0000	*638	.0000	17
1	.2003	*835	.0013	24	22.96	*956	.1990	*810	.0638	*265	.0017	31
2	.2838	*644	.0037	32	32.52	*738	.2800	*612	.0903	*205	.0048	39
3	.3482	*545	.0069	36	39.90	*625	.3412	*507	.1108	*174	.0087	47
4	.4027	*483	.0105	42	46.15	*553	.3919	*440	.1282	*154	.0134	53
.05	.4510	*439	.0147	45	51.68°	*504	.4359	*391	.1436	*139	.0187	58
6	.4949	*406	.0192	50	56.72	*465	.4750	*353	.1575	*130	.0245	63
7	.5355	*380	.0242	52	61.37	*435	.5103	*323	.1705	121	.0308	67
8	.5735	*359	.0294	56	65.72	*411	.5426	*298	.1826	114	.0375	71
9	.6094	*341	.0350	59	69.83	*391	.5724	*276	.1940	108	.0446	74
.10	.6435	*326	.0409	61	73.74°	*374	.6000	*258	.2048	104	.0520	79
1	.6761	*314	.0470	64	77.48	*359	.6258	*241	.2152	100	.0599	81
2	.7075	*302	.0534	66	81.07	*347	.6499	*227	.2252	96	.0680	84
3	.7377	*293	.0600	68	84.54	*335	.6726	*214	.2348	93	.0764	87
4	.7670	*284	.0668	71	87.89	*326	.6940	*201	.2441	91	.0851	90
.15	.7954	276	.0739	72	91.15°	316	.7141	*191	.2532	88	.0941	92
6	.8230	270	.0811	74	94.31	309	.7332	*181	.2620	86	.1033	94
7	.8500	263	.0885	76	97.40	302	.7513	*171	.2706	83	.1127	97
8	.8763	258	.0961	78	100.42	295	.7684	162	.2789	82	.1224	99
9	.9021	252	.1039	79	103.37	289	.7846	154	.2871	81	.1323	101
.20	0.9273	248	.1118	81	106.26°	284	.8000	146	.2952	79	.1424	103
1	0.9521	243	.1199	82	109.10	279	.8146	139	.3031	77	.1527	104
2	0.9764	240	.1281	84	111.89	274	.8285	132	.3108	76	.1631	106
3	1.0004	235	.1365	84	114.63	271	.8417	125	.3184	75	.1737	109
4	1.0239	233	.1449	86	117.34	266	.8542	118	.3259	74	.1846	109
.25	1.0472	229	.1535	88	120.00°	263	.8660	113	.3333	73	.1955	111
6	1.0701	227	.1623	88	122.63	260	.8773	106	.3406	72	.2066	112
7	1.0928	224	.1711	89	125.23	256	.8879	101	.3478	72	.2178	114
8	1.1152	222	.1800	90	127.79	254	.8980	95	.3550	70	.2292	115
9	1.1374	219	.1890	92	130.33	251	.9075	90	.3620	70	.2407	116
.30	1.1593	217	.1982	92	132.84°	249	.9165	85	.3690	69	.2523	117
1	1.1810	215	.2074	93	135.33	247	.9250	80	.3759	69	.2640	119
2	1.2025	214	.2167	93	137.80	245	.9330	74	.3828	68	.2759	119
3	1.2239	212	.2260	95	140.25	242	.9404	70	.3896	67	.2878	120
4	1.2451	210	.2355	95	142.67	241	.9474	65	.3963	67	.2998	121
.35	1.2661	209	.2450	96	145.08°	240	.9539	61	.4030	67	.3119	122
6	1.2870	208	.2546	96	147.48	238	.9600	56	.4097	66	.3241	123
7	1.3078	206	.2642	97	149.86	237	.9656	52	.4163	66	.3364	123
8	1.3284	206	.2739	97	152.23	235	.9708	47	.4229	65	.3487	124
9	1.3490	204	.2836	98	154.58	235	.9755	43	.4294	65	.3611	124
.40	1.3694	204	.2934	98	156.93°	233	.9798	39	.4359	65	.3735	125
1	1.3898	203	.3032	98	159.26	233	.9837	34	.4424	65	.3860	126
2	1.4101	202	.3130	99	161.59	231	.9871	31	.4489	64	.3986	126
3	1.4303	202	.3229	99	163.90	232	.9902	26	.4553	64	.4112	126
4	1.4505	201	.3328	100	166.22	230	.9928	22	.4617	64	.4238	126
.45	1.4706	201	.3428	99	168.52°	230	.9950	18	.4681	64	.4364	127
6	1.4907	201	.3527	100	170.82	230	.9968	14	.4745	64	.4491	127
7	1.5108	200	.3627	100	173.12	230	.9982	10	.4809	64	.4618	127
8	1.5308	200	.3727	100	175.42	229	.9992	6	.4873	63	.4745	128
9	1.5508	200	.3827	100	177.71	229	.9998	2	.4936	64	.4873	127
.50	1.5708		.3927		180.00°		1.0000		.5000		.5000	

* Interpolation may be inaccurate at these points.

VOLUMES OF SPHERES BY HUNDREDTHS

D	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	.5236	.5395	.5556	.5722	.5890	.6061	.6236	.6414	.6596	.6781	173
.1	.6969	.7161	.7356	.7555	.7757	.7963	.8173	.8386	.8603	.8823	208
.2	.9048	.9276	.9508	.9743	.9983	1.0227					236
.2						1.023	1.047	1.073	1.098	1.124	25
.3	1.150	1.177	1.204	1.232	1.260	1.288	1.317	1.346	1.376	1.406	29
.4	1.437	1.468	1.499	1.531	1.563	1.596	1.630	1.663	1.697	1.732	33
1.5	1.767	1.803	1.839	1.875	1.912	1.950	1.988	2.026	2.065	2.105	38
.6	2.145	2.185	2.226	2.268	2.310	2.352	2.395	2.439	2.483	2.527	43
.7	2.572	2.618	2.664	2.711	2.758	2.806	2.855	2.903	2.953	3.003	48
.8	3.054	3.105	3.157	3.209	3.262	3.315	3.369	3.424	3.479	3.535	54
.9	3.591	3.648	3.706	3.764	3.823	3.882	3.942	4.003	4.064	4.126	60
2.0	4.189	4.252	4.316	4.380	4.445	4.511	4.577	4.644	4.712	4.780	66
.1	4.849	4.919	4.989	5.060	5.131	5.204	5.277	5.350	5.425	5.500	73
.2	5.575	5.652	5.729	5.806	5.885	5.964	6.044	6.125	6.206	6.288	80
.3	6.371	6.454	6.538	6.623	6.709	6.795	6.882	6.970	7.059	7.148	87
.4	7.238	7.329	7.421	7.513	7.606	7.700	7.795	7.890	7.986	8.083	94
2.5	8.181	8.280	8.379	8.479	8.580	8.682	8.785	8.888	8.992	9.097	102
.6	9.203	9.309	9.417	9.525	9.634	9.744	9.855	9.966	10.079		110
.6									10.08	10.19	11
.7	10.31	10.42	10.54	10.65	10.77	10.89	11.01	11.13	11.25	11.37	12
.8	11.49	11.62	11.74	11.87	11.99	12.12	12.25	12.38	12.51	12.64	13
.9	12.77	12.90	13.04	13.17	13.31	13.44	13.58	13.72	13.86	14.00	14
3.0	14.14	14.28	14.42	14.57	14.71	14.86	15.00	15.15	15.30	15.45	15
.1	15.60	15.75	15.90	16.06	16.21	16.37	16.52	16.68	16.84	17.00	16
.2	17.16	17.32	17.48	17.64	17.81	17.97	18.14	18.31	18.48	18.65	17
.3	18.82	18.99	19.16	19.33	19.51	19.68	19.86	20.04	20.22	20.40	18
.4	20.58	20.76	20.94	21.13	21.31	21.50	21.69	21.88	22.07	22.26	19
3.5	22.45	22.64	22.84	23.03	23.23	23.43	23.62	23.82	24.02	24.23	20
.6	24.43	24.63	24.84	25.04	25.25	25.46	25.67	25.88	26.09	26.31	21
.7	26.52	26.74	26.95	27.17	27.39	27.61	27.83	28.06	28.28	28.50	22
.8	28.73	28.96	29.19	29.42	29.65	29.88	30.11	30.35	30.58	30.82	23
.9	31.06	31.30	31.54	31.78	32.02	32.27	32.52	32.76	33.01	33.26	25
4.0	33.51	33.76	34.02	34.27	34.53	34.78	35.04	35.30	35.56	35.82	26
.1	36.09	36.35	36.62	36.88	37.15	37.42	37.69	37.97	38.24	38.52	27
.2	38.79	39.07	39.35	39.63	39.91	40.19	40.48	40.76	41.05	41.34	28
.3	41.63	41.92	42.21	42.51	42.80	43.10	43.40	43.70	44.00	44.30	30
.4	44.60	44.91	45.21	45.52	45.83	46.14	46.45	46.77	47.08	47.40	31
4.5	47.71	48.03	48.35	48.67	49.00	49.32	49.65	49.97	50.30	50.63	33
.6	50.97	51.30	51.63	51.97	52.31	52.65	52.99	53.33	53.67	54.02	34
.7	54.36	54.71	55.06	55.41	55.76	56.12	56.47	56.83	57.19	57.54	35
.8	57.91	58.27	58.63	59.00	59.37	59.73	60.10	60.48	60.85	61.22	37
.9	61.60	61.98	62.36	62.74	63.12	63.51	63.89	64.28	64.67	65.06	38

Explanation of Table of Volumes of Spheres (pp. 36–37).

Moving the decimal point **one** place in column D is equivalent to moving it **three** places in the body of the table. (D = diameter.)

$$\text{Volume of sphere} = \frac{\pi}{6} \times (\text{diam.}^3) = 0.523599 \times (\text{diam.}^3)$$

Conversely,

$$\text{Diam.} = \sqrt[3]{\frac{6}{\pi}} \times \sqrt[3]{\text{volume}} = 1.240701 \times \sqrt[3]{\text{volume}}$$

VOLUMES OF SPHERES *(continued)*

D	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	65.45	65.84	66.24	66.64	67.03	67.43	67.83	68.24	68.64	69.05	40
.1	69.46	69.87	70.28	70.69	71.10	71.52	71.94	72.36	72.78	73.20	42
.2	73.62	74.05	74.47	74.90	75.33	75.77	76.20	76.64	77.07	77.51	43
.3	77.95	78.39	78.84	79.28	79.73	80.18	80.63	81.08	81.54	81.99	45
.4	82.45	82.91	83.37	83.83	84.29	84.76	85.23	85.70	86.17	86.64	47
5.5	87.11	87.59	88.07	88.55	89.03	89.51	90.00	90.48	90.97	91.46	48
.6	91.95	92.45	92.94	93.44	93.94	94.44	94.94	95.44	95.95	96.46	50
.7	96.97	97.48	97.99	98.51	99.02	99.54	100.06				52
.7							100.1	100.6	101.1	101.6	5
.8	102.2	102.7	103.2	103.8	104.3	104.8	105.4	105.9	106.4	107.0	5
.9	107.5	108.1	108.6	109.2	109.7	110.3	110.9	111.4	112.0	112.5	6
6.0	113.1	113.7	114.2	114.8	115.4	115.9	116.5	117.1	117.7	118.3	6
.1	118.8	119.4	120.0	120.6	121.2	121.8	122.4	123.0	123.6	124.2	
.2	124.8	125.4	126.0	126.6	127.2	127.8	128.4	129.1	129.7	130.3	
.3	130.9	131.5	132.2	132.8	133.4	134.1	134.7	135.3	136.0	136.6	
.4	137.3	137.9	138.5	139.2	139.8	140.5	141.2	141.8	142.5	143.1	7
6.5	143.8	144.5	145.1	145.8	146.5	147.1	147.8	148.5	149.2	149.8	
.6	150.5	151.2	151.9	152.6	153.3	154.0	154.7	155.4	156.1	156.8	
.7	157.5	158.2	158.9	159.6	160.3	161.0	161.7	162.5	163.2	163.9	
.8	164.6	165.4	166.1	166.8	167.6	168.3	169.0	169.8	170.5	171.3	
.9	172.0	172.8	173.5	174.3	175.0	175.8	176.5	177.3	178.1	178.8	8
7.0	179.6	180.4	181.1	181.9	182.7	183.5	184.3	185.0	185.8	186.6	
.1	187.4	188.2	189.0	189.8	190.6	191.4	192.2	193.0	193.8	194.6	
.2	195.4	196.2	197.1	197.9	198.7	199.5	200.4	201.2	202.0	202.9	
.3	203.7	204.5	205.4	206.2	207.1	207.9	208.8	209.6	210.5	211.3	
.4	212.2	213.0	213.9	214.8	215.6	216.5	217.4	218.3	219.1	220.0	9
7.5	220.9	221.8	222.7	223.6	224.4	225.3	226.2	227.1	228.0	228.9	
.6	229.8	230.8	231.7	232.6	233.5	234.4	235.3	236.3	237.2	238.1	
.7	239.0	240.0	240.9	241.8	242.8	243.7	244.7	245.6	246.6	247.5	
.8	248.5	249.4	250.4	251.4	252.3	253.3	254.3	255.2	256.2	257.2	10
.9	258.2	259.1	260.1	261.1	262.1	263.1	264.1	265.1	266.1	267.1	
8.0	268.1	269.1	270.1	271.1	272.1	273.1	274.2	275.2	276.2	277.2	
.1	278.3	279.3	280.3	281.4	282.4	283.4	284.5	285.5	286.6	287.6	
.2	288.7	289.8	290.8	291.9	292.9	294.0	295.1	296.2	297.2	298.3	11
.3	299.4	300.5	301.6	302.6	303.7	304.8	305.9	307.0	308.1	309.2	
.4	310.3	311.4	312.6	313.7	314.8	315.9	317.0	318.2	319.3	320.4	
8.5	321.6	322.7	323.8	325.0	326.1	327.3	328.4	329.6	330.7	331.9	
.6	333.0	334.2	335.4	336.5	337.7	338.9	340.1	341.2	342.4	343.6	12
.7	344.8	346.0	347.2	348.4	349.6	350.8	352.0	353.2	354.4	355.6	
.8	356.8	358.0	359.3	360.5	361.7	362.9	364.2	365.4	366.6	367.9	
.9	369.1	370.4	371.6	372.9	374.1	375.4	376.6	377.9	379.2	380.4	13
9.0	381.7	383.0	384.3	385.5	386.8	388.1	389.4	390.7	392.0	393.3	
.1	394.6	395.9	397.2	398.5	399.8	401.1	402.4	403.7	405.1	406.4	
.2	407.7	409.1	410.4	411.7	413.1	414.4	415.7	417.1	418.4	419.8	
.3	421.2	422.5	423.9	425.2	426.6	428.0	429.4	430.7	432.1	433.5	14
.4	434.9	436.3	437.7	439.1	440.5	441.9	443.3	444.7	446.1	447.5	
9.5	448.9	450.3	451.8	453.2	454.6	456.0	457.5	458.9	460.4	461.8	
.6	463.2	464.7	466.1	467.6	469.1	470.5	472.0	473.5	474.9	476.4	15
.7	477.9	479.4	480.8	482.3	483.8	485.3	486.8	488.3	489.8	491.3	
.8	492.8	494.3	495.8	497.3	498.9	500.4	501.9	503.4	505.0	506.5	16
9	508.0	509.6	511.1	512.7	514.2	515.8	517.3	518.9	520.5	522.0	
10.0	523.6										

Moving the decimal point ONE place in D requires moving it THREE places in body of table (see p. 36).

SEGMENTS OF SPHERES

(h = height of segment; D = diam. of sphere)

$\frac{h}{D}$	$\frac{\text{Vol. segm.}}{D^3}$	Diff.	$\frac{\text{Vol. segm.}}{\text{Vol. sphere}}$	Diff.
0.00	0.0000	2	0.0000	3
1	0.0002	4	0.0003	9
2	0.0006	8	0.0012	14
3	0.0014	10	0.0026	21
4	0.0024	14	0.0047	26
0.05	0.0038	16	0.0073	31
6	0.0054	19	0.0104	36
7	0.0073	22	0.0140	42
8	0.0095	25	0.0182	46
9	0.0120	27	0.0228	52
0.10	0.0147	29	0.0280	56
1	0.0176	32	0.0336	61
2	0.0208	34	0.0397	66
3	0.0242	37	0.0463	70
4	0.0279	39	0.0533	74
0.15	0.0318	41	0.0607	79
6	0.0359	44	0.0686	83
7	0.0403	45	0.0769	86
8	0.0448	47	0.0855	91
9	0.0495	50	0.0946	94
0.20	0.0545	51	0.1040	98
1	0.0596	53	0.1138	101
2	0.0649	55	0.1239	105
3	0.0704	56	0.1344	108
4	0.0760	58	0.1452	110
0.25	0.0818	60	0.1562	114
6	0.0878	61	0.1676	117
7	0.0939	63	0.1793	120
8	0.1002	64	0.1913	122
9	0.1066	65	0.2035	125
0.30	0.1131	67	0.2160	127
1	0.1198	67	0.2287	130
2	0.1265	69	0.2417	131
3	0.1334	70	0.2548	134
4	0.1404	71	0.2682	135
0.35	0.1475	72	0.2817	138
6	0.1547	73	0.2955	139
7	0.1620	74	0.3094	141
8	0.1694	74	0.3235	142
9	0.1768	75	0.3377	143
0.40	0.1843	76	0.3520	145
1	0.1919	76	0.3665	145
2	0.1995	77	0.3810	147
3	0.2072	77	0.3957	147
4	0.2149	78	0.4104	148
0.45	0.2227	78	0.4252	149
6	0.2305	78	0.4401	150
7	0.2383	78	0.4551	149
8	0.2461	78	0.4700	150
9	0.2539	79	0.4850	150
0.50	0.2618		0.5000	

Explanation of Table on this page

Given, h = height of segment,
D = diam. of sphere.

To find the volume of the segment, form the ratio h/D and find from the table the value of (vol./D^3); then, by a simple multiplication,

vol. segment = $D^3 \times$ (vol./D^3)

The table gives also the ratio of the volume of the segment to the entire volume of the sphere.

NOTE. Area of zone = $\pi \times h \times D$. (Use Table of Multiples of π, p. 28)

Explanation of Table on p. 34

Given, h = height of segment,
c = chord.

To find the diam. of the circle, the length of arc, or the area of the segment, form the ratio h/c, and find from the table the value of (diam./c), (arc/c), or (area/hc); then, by a simple multiplication,

diam. $= c \times$ (diam./c),
arc $= c \times$ (arc/c),
area $= h \times c \times$ (area/hc).

The table gives also the angle subtended at the center, and the ratio of h to D. See p. 106.

Explanation of Table on p. 35

Given, h = height of segment,
D = diam. of circle.

To find the chord, the length of arc, or the area of the segment, form the ratio h/D, and find from the table the value of (chord/D), (arc/D), or (area/D^2); then, by a simple multiplication,

chord $= D \times$ (chord/D),
arc $= D \times$ (arc/D),
area $= D^2 \times$ (area/D^2).

The table gives also the angle subtended at the center, the ratio of the arc of the segment to the whole circumference, and the ratio of the area of the segment to the area of the whole circle. See p. 106.

NOTE. Vol. segm. $= \frac{1}{6}\pi h^2 (3D-2h)$.

REGULAR POLYGONS

n = number of sides;
$v = 360°/n$ = angle subtended at the center by one side;

$$a = \text{length of one side} = R\left(2\sin\frac{v}{2}\right) = r\left(2\tan\frac{v}{2}\right);$$

$$R = \text{radius of circumscribed circle} = a\left(\tfrac{1}{2}\csc\frac{v}{2}\right) = r\left(\sec\frac{v}{2}\right);$$

$$r = \text{radius of inscribed circle} = R\left(\cos\frac{v}{2}\right) = a\left(\tfrac{1}{2}\cot\frac{v}{2}\right);$$

$$\text{Area} = a^2\left(\tfrac{1}{4}n\cot\frac{v}{2}\right) = R^2\left(\tfrac{1}{2}n\sin v\right) = r^2\left(n\tan\frac{v}{2}\right).$$

n	v	$\frac{\text{Area}}{a^2}$	$\frac{\text{Area}}{R^2}$	$\frac{\text{Area}}{r^2}$	$\frac{R}{a}$	$\frac{R}{r}$	$\frac{a}{R}$	$\frac{a}{r}$	$\frac{r}{R}$	$\frac{r}{a}$
3	120°	0.4330	1.299	5.196	0.5774	2.000	1.732	3.464	0.5000	0.2887
4	90°	1.000	2.000	4.000	0.7071	1.414	1.414	2.000	0.7071	0.5000
5	72°	1.721	2.378	3.633	0.8507	1.236	1.176	1.453	0.8090	0.6882
6	60°	2.598	2.598	3.464	1.0000	1.155	1.000	1.155	0.8660	0.8660
7	51°.43	3.634	2.736	3.371	1.152	1.110	0.8678	0.9631	0.9010	1.038
8	45°	4.828	2.828	3.314	1.307	1.082	0.7654	0.8284	0.9239	1.207
9	40°	6.182	2.893	3.276	1.462	1.064	0.6840	0.7279	0.9397	1.374
10	36°	7.694	2.939	3.249	1.618	1.052	0.6180	0.6498	0.9511	1.539
12	30°	11.20	3.000	3.215	1.932	1.035	0.5176	0.5359	0.9659	1.866
15	24°	17.64	3.051	3.188	2.405	1.022	0.4158	0.4251	0.9781	2.352
16	22°.50	20.11	3.062	3.183	2.563	1.020	0.3902	0.3978	0.9808	2.514
20	18°	31.57	3.090	3.168	3.196	1.013	0.3129	0.3168	0.9877	3.157
24	15°	45.58	3.106	3.160	3.831	1.009	0.2611	0.2633	0.9914	3.798
32	11°.25	81.23	3.121	3.152	5.101	1.005	0.1960	0.1970	0.9952	5.077
48	7°.50	183.1	3.133	3.146	7.645	1.002	0.1308	0.1311	0.9979	7.629
64	5°.625	325.7	3.137	3.144	10.19	1.001	0.0981	0.0983	0.9988	10.18

BINOMIAL COEFFICIENTS

(For table giving binomial coefficients for fractional values of n, see p. 116).

$$(n)_0 = 1;\ (n)_1 = n;\ (n)_2 = \frac{n(n-1)}{1 \times 2};\ (n)_3 = \frac{n(n-1)(n-2)}{1 \times 2 \times 3};\ \text{etc.; in general,}$$

$$(n)_r = \frac{n(n-1)(n-2)\ .\ .\ .\ (n-[r-1])}{1 \times 2 \times 3\ .\ .\ .\ \times r}.\quad \text{Another notation: } \binom{n}{r} = (n)_r.$$

n	$(n)_0$	$(n)_1$	$(n)_2$	$(n)_3$	$(n)_4$	$(n)_5$	$(n)_6$	$(n)_7$	$(n)_8$	$(n)_9$	$(n)_{10}$	$(n)_{11}$	$(n)_{12}$	$(n)_{13}$
1	1	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
2	1	2	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
3	1	3	3	1	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
4	1	4	6	4	1	.....	.....	.....	.....	.....	.....	.....	.....	.....
5	1	5	10	10	5	1	.....	.....	.....	.....	.....	.....	.....	.....
6	1	6	15	20	15	6	1	.....	.....	.....	.....	.....	.....	.....
7	1	7	21	35	35	21	7	1	.....	.....	.....	.....	.....	.....
8	1	8	28	56	70	56	28	8	1	.....	.....	.....	.....	.....
9	1	9	36	84	126	126	84	36	9	1	.....	.....	.....	.....
10	1	10	45	120	210	252	210	120	45	10	1	.....	.....	.....
11	1	11	55	165	330	462	462	330	165	55	11	1	.....	.....
12	1	12	66	220	495	792	924	792	495	220	66	12	1	.....
13	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13	1
14	1	14	91	364	1001	2002	3003	3432	3003	2002	1001	364	91	14
15	1	15	105	455	1365	3003	5005	6435	6435	5005	3003	1365	455	105

For $n = 14$, $(n)_{14} = 1$; for $n = 15$, $(n)_{14} = 15$, and $(n)_{15} = 1$.

COMMON LOGARITHMS (*special table*)

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.00	0.0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	4
1.01	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082	
1.02	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124	
1.03	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166	
1.04	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208	
1.05	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249	
1.06	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290	
1.07	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330	
1.08	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370	
1.09	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410	
1.10	0.0414	0418	0422	0426	0430	0434	0438	0441	0445	0449	
1.11	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488	
1.12	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527	
1.13	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565	
1.14	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603	
1.15	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641	
1.16	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678	
1.17	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715	
1.18	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752	
1.19	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788	
1.20	0.0792	0795	0799	0803	0806	0810	0813	0817	0821	0824	
1.21	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860	
1.22	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896	
1.23	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931	
1.24	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966	
1.25	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000	3
1.26	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035	
1.27	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069	
1.28	1072	1075	1079	1082	1086	1089	1092	1096	1099	1103	
1.29	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136	
1.30	0.1139	1143	1146	1149	1153	1156	1159	1163	1166	1169	
1.31	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202	
1.32	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235	
1.33	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268	
1.34	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300	
1.35	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332	
1.36	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364	
1.37	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396	
1.38	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427	
1.39	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458	
1.40	0.1461	1464	1467	1471	1474	1477	1480	1483	1486	1489	
1.41	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520	
1.42	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550	
1.43	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581	
1.44	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611	
1.45	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641	
1.46	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670	
1.47	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	
1.48	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729	
1.49	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	

Moving the decimal point n places to the right [or left] in the number requires adding $+ n$ [or $- n$] in the body of the table (see p. 42).

COMMON LOGARITHMS (*special table, continued*)

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.50	0.1761	1764	1767	1770	1772	1775	1778	1781	1784	1787	3
1.51	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816	
1.52	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844	
1.53	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872	
1.54	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901	
1.55	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928	
1.56	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956	
1.57	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984	
1.58	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011	
1.59	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038	
1.60	0.2041	2044	2047	2049	2052	2055	2057	2060	2063	2066	
1.61	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092	
1.62	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119	
1.63	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146	
1.64	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172	
1.65	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198	
1.66	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225	
1.67	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251	
1.68	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276	
1.69	2279	2281	2284	2287	2289	2292	2294	2297	2299	2302	
1.70	0.2304	2307	2310	2312	2315	2317	2320	2322	2325	2327	
1.71	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353	
1.72	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378	
1.73	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403	
1.74	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428	2
1.75	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453	
1.76	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477	
1.77	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502	
1.78	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526	
1.79	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550	
1.80	0.2553	2555	2558	2560	2562	2565	2567	2570	2572	2574	
1.81	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598	
1.82	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622	
1.83	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646	
1.84	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669	
1.85	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693	
1.86	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716	
1.87	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739	
1.88	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762	
1.89	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785	
1.90	0.2788	2790	2792	2794	2797	2799	2801	2804	2806	2808	
1.91	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831	
1.92	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853	
1.93	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876	
1.94	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898	
1.95	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920	
1.96	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942	
1.97	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964	
1.98	2967	2969	2971	2973	2975	2978	2980	2982	2984	2986	
1.99	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008	

COMMON LOGARITHMS

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	0.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	See pages 40–41
1.1	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	
1.2	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	
1.3	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	
1.4	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	
1.5	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	
1.6	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	
1.7	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	
1.8	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	
1.9	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	
2.0	0.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
2.1	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
2.2	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
2.3	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
2.4	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	17
2.5	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
2.6	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
2.7	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
2.8	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
2.9	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
3.0	0.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
3.1	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
3.2	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
3.3	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
3.4	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
3.5	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
3.6	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
3.7	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
3.8	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
3.9	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
4.0	0.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
4.1	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
4.2	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
4.3	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
4.4	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
4.5	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
4.6	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	10
4.7	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
4.8	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
4.9	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9

$\log \pi = 0.4971$ $\quad \log \pi/2 = 0.1961$ $\quad \log \pi^2 = 0.9943$ $\quad \log \sqrt{\pi} = 0.2486$
$\log e = 0.4343$ $\quad \log (0.4343) = 0.6378 - 1$

These two pages give the common logarithms of numbers between 1 and 10, correct to four places. Moving the decimal point n places to the right [or left] in the number is equivalent to adding n [or $-n$] to the logarithm. Thus, $\log 0.017453 = 0.2419 - 2$, which may also be written $\bar{2}.2419$ or $8.2419 - 10$. See p. 91. Graphs, p. 174.

$$\log (ab) = \log a + \log b \qquad \log (a^N) = N \log a$$

$$\log \left(\frac{a}{b}\right) = \log a - \log b \qquad \log \left(\sqrt[N]{a}\right) = \frac{1}{N} \log a$$

COMMON LOGARITHMS (*continued*)

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	0.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
5.1	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
5.2	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
5.3	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
5.4	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
5.5	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
5.6	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
5.7	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
5.8	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
5.9	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
6.0	0.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
6.1	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
6.2	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
6.3	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
6.4	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
6.5	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
6.6	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
6.7	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
6.8	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
6.9	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
7.0	0.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
7.1	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
7.2	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
7.3	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
7.4	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
7.5	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
7.6	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
7.7	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
7.8	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
7.9	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
8.0	0.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
8.1	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
8.2	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
8.3	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
8.4	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
8.5	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
8.6	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
8.7	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
8.8	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
8.9	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
9.0	0.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
9.1	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
9.2	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
9.3	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
9.4	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
9.5	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
9.6	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	4
9.7	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
9.8	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
9.9	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

DEGREES AND MINUTES EXPRESSED IN RADIANS (See also p. 69)

Degrees						Hundredths				Minutes	
1°	.0175	61°	1.0647	121°	2.1118	0°.01	.0002	0°.51	.0089	1′	.0003
2	.0349	2	1.0821	2	2.1293	2	.0003	2	.0091	2′	.0006
3	.0524	3	1.0996	3	2.1468	3	.0005	3	.0093	3′	.0009
4	.0698	4	1.1170	4	2.1642	4	.0007	4	.0094	4′	.0012
5°	.0873	65°	1.1345	125°	2.1817	.05	.0009	.55	.0096	5′	.0015
6	.1047	6	1.1519	6	2.1991	6	.0010	6	.0098	6′	.0017
7	.1222	7	1.1694	7	2.2166	7	.0012	7	.0099	7′	.0020
8	.1396	8	1.1868	8	2.2340	8	.0014	8	.0101	8′	.0023
9	.1571	9	1.2043	9	2.2515	9	.0016	9	.0103	9′	.0026
10°	.1745	70°	1.2217	130°	2.2689	0°.10	.0017	0°.60	.0105	10′	.0029
1	.1920	1	1.2392	1	2.2864	1	.0019	1	.0106	11′	.0032
2	.2094	2	1.2566	2	2.3038	2	.0021	2	.0108	12′	.0035
3	.2269	3	1.2741	3	2.3213	3	.0023	3	.0110	13′	.0038
4	.2443	4	1.2915	4	2.3387	4	.0024	4	.0112	14′	.0041
15°	.2618	75°	1.3090	135°	2.3562	.15	.0026	.65	.0113	15′	0044
6	.2793	6	1.3265	6	2.3736	6	.0028	6	.0115	16′	.0047
7	.2967	7	1.3439	7	2.3911	7	.0030	7	.0117	17′	.0049
8	.3142	8	1.3614	8	2.4086	8	.0031	8	.0119	18′	.0052
9	.3316	9	1.3788	9	2.4260	9	.0033	9	.0120	19′	.0055
20°	.3491	80°	1.3963	140°	2.4435	0°.20	.0035	0°.70	.0122	20′	.0058
1	.3665	1	1.4137	1	2.4609	1	.0037	1	.0124	21′	.0061
2	.3840	2	1.4312	2	2.4784	2	.0038	2	.0126	22′	.0064
3	.4014	3	1.4486	3	2.4958	3	.0040	3	.0127	23′	.0067
4	.4189	4	1.4661	4	2.5133	4	.0042	4	.0129	24′	.0070
25°	.4363	85°	1.4835	145°	2.5307	.25	.0044	.75	.0131	25′	0073
6	.4538	6	1.5010	6	2.5482	6	.0045	6	.0133	26′	.0076
7	.4712	7	1.5184	7	2.5656	7	.0047	7	.0134	27′	.0079
8	.4887	8	1.5359	8	2.5831	8	.0049	8	.0136	28′	.0081
9	.5061	9	1.5533	9	2.6005	9	.0051	9	.0138	29′	0084
30°	.5236	90°	1.5708	150°	2.6180	0°.30	.0052	0°.80	.0140	30′	.0087
1	.5411	1	1.5882	1	2.6354	1	.0054	1	.0141	31′	.0090
2	.5585	2	1.6057	2	2.6529	2	.0056	2	.0143	32′	.0093
3	.5760	3	1.6232	3	2.6704	3	.0058	3	.0145	33′	.0096
4	.5934	4	1.6406	4	2.6878	4	.0059	4	.0147	34′	.0099
35°	.6109	95°	1.6581	155°	2.7053	35	.0061	.85	.0148	35′	.0102
6	.6283	6	1.6755	6	2.7227	6	.0063	6	.0150	36′	.0105
7	.6458	7	1.6930	7	2.7402	7	.0065	7	.0152	37′	.0108
8	.6632	8	1.7104	8	2.7576	8	.0066	8	.0154	38′	.0111
9	.6807	9	1.7279	9	2.7751	9	.0068	9	.0155	39′	.0113
40°	.6981	100°	1.7453	160°	2.7925	0°.40	.0070	0°.90	.0157	40′	.0116
1	.7156	1	1.7628	1	2.8100	1	.0072	1	.0159	41′	.0119
2	.7330	2	1.7802	2	2.8274	2	.0073	2	.0161	42′	.0122
3	.7505	3	1.7977	3	2.8449	3	.0075	3	.0162	43′	.0125
4	.7679	4	1.8151	4	2.8623	4	.0077	4	.0164	44′	.0128
45°	.7854	105°	1.8326	165°	2.8798	.45	.0079	.95	.0166	45′	.0131
6	.8029	6	1.8500	6	2.8972	6	.0080	6	.0168	46′	.0134
7	.8203	7	1.8675	7	2.9147	7	.0082	7	.0169	47′	.0137
8	.8378	8	1.8850	8	2.9322	8	.0084	8	.0171	48′	.0140
9	.8552	9	1.9024	9	2.9496	9	.0086	9	.0173	49′	.0143
50°	.8727	110°	1.9199	170°	2.9671	0°.50	.0087	1°.00	.0175	50′	.0145
1	.8901	1	1.9373	1	2.9845					51′	.0148
2	.9076	2	1.9548	2	3.0020					52′	.0151
3	.9250	3	1.9722	3	3.0194					53′	.0154
4	.9425	4	1.9897	4	3.0369					54′	.0157
55°	.9599	115°	2.0071	175°	3.0543					55′	.0160
6	.9774	6	2.0246	6	3.0718					56′	.0163
7	.9948	7	2.0420	7	3.0892					57′	.0166
8	1.0123	8	2.0595	8	3.1067					58′	.0169
9	1.0297	9	2.0769	9	3.1241					59′	.0172
60°	1.0472	120°	2.0944	180°	3.1416					60′	.0175

Arc 1° = 0.0174533 Arc 1′ = 0.000290888 Arc 1″ = 0.00000484814
1 radian = 57°.295780 = 57° 17′.7468 = 57° 17′ 44″.806

RADIANS EXPRESSED IN DEGREES

0.01	0°.57	.64	36°.67	1.27	72°.77	1.90	108°.86	2.53	144°.96
2	1°.15	.65	37°.24	8	73°.34	1	109°.43	4	145°.53
3	1°.72	6	37°.82	9	73°.91	2	110°.01	2.55	146°.10
4	2°.29	7	38°.39	1.30	74°.48	3	110°.58	6	146°.68
.05	2°.86	8	38°.96	1	75°.06	4	111°.15	7	147°.25
6	3°.44	9	39°.53	2	75°.63	1.95	111°.73	8	147°.82
7	4°.01	.70	40°.11	3	76°.20	6	112°.30	9	148°.40
8	4°.58	1	40°.68	4	76°.78	7	112°.87	2.60	148°.97
9	5°.16	2	41°.25	1.35	77°.35	8	113°.45	1	149°.54
.10	5°.73	3	41°.83	6	77°.92	9	114°.02	2	150°.11
1	6°.30	4	42°.40	7	78°.50	2.00	114°.59	3	150°.69
2	6°.88	.75	42°.97	8	79°.07	1	115°.16	4	151°.26
3	7°.45	6	43°.54	9	79°.64	2	115°.74	2.65	151°.83
4	8°.02	7	44°.12	1.40	80°.21	3	116°.31	6	152°.41
.15	8°.59	8	44°.69	1	80°.79	4	116°.88	7	152°.98
6	9°.17	9	45°.26	2	81°.36	2.05	117°.46	8	153°.55
7	9°.74	.80	45°.84	3	81°.93	6	118°.03	9	154°.13
8	10°.31	1	46°.41	4	82°.51	7	118°.60	2.70	154°.70
9	10°.89	2	46°.98	1.45	83°.08	8	119°.18	1	155°.27
.20	11°.46	3	47°.56	6	83°.65	9	119°.75	2	155°.84
1	12°.03	4	48°.13	7	84°.22	2.10	120°.32	3	156°.42
2	12°.61	.85	48°.70	8	84°.80	1	120°.89	4	156°.99
3	13°.18	6	49°.27	9	85°.37	2	121°.47	2.75	157°.56
4	13°.75	7	49°.85	1.50	85°.94	3	122°.04	6	158°.14
.25	14°.32	8	50°.42	1	86°.52	4	122°.61	7	158°.71
6	14°.90	9	50°.99	2	87°.09	2.15	123°.19	8	159°.28
7	15°.47	.90	51°.57	3	87°.66	6	123°.76	9	159°.86
8	16°.04	1	52°.14	4	88°.24	7	124°.33	2.80	160°.43
9	16°.62	2	52°.71	1.55	88°.81	8	124°.90	1	161°.00
.30	17°.19	3	53°.29	6	89°.38	9	125°.48	2	161°.57
1	17°.76	4	53°.86	7	89°.95	2.20	126°.05	3	162°.15
2	18°.33	.95	54°.43	8	90°.53	1	126°.62	4	162°.72
3	18°.91	6	55°.00	9	91°.10	2	127°.20	2.85	163°.29
4	19°.48	7	55°.58	1.60	91°.67	3	127°.77	6	163°.87
.35	20°.05	8	56°.15	1	92°.25	4	128°.34	7	164°.44
6	20°.63	9	56°.72	2	92°.82	2.25	128°.92	8	165°.01
7	21°.20	1.00	57°.30	3	93°.39	6	129°.49	9	165°.58
8	21°.77	1	57°.87	4	93°.97	7	130°.06	2.90	166°.16
9	22°.35	2	58°.44	1.65	94°.54	8	130°.63	1	166°.73
.40	22°.92	3	59°.01	6	95°.11	9	131°.21	2	167°.30
1	23°.49	4	59°.59	7	95°.68	2.30	131°.78	3	167°.88
2	24°.06	1.05	60°.16	8	96°.26	1	132°.35	4	168°.45
3	24°.64	6	60°.73	9	96°.83	2	132°.93	2.95	169°.02
4	25°.21	7	61°.31	1.70	97°.40	3	133°.50	6	169°.60
.45	25°.78	8	61°.88	1	97°.98	4	134°.07	7	170°.17
6	26°.36	9	62°.45	2	98°.55	2.35	134°.65	8	170°.74
7	26°.93	1.10	63°.03	3	99°.12	6	135°.22	9	171°.31
8	27°.50	1	63°.60	4	99°.69	7	135°.79	3.00	171°.89
9	28°.07	2	64°.17	1.75	100°.27	8	136°.36	1	172°.46
.50	28°.65	3	64°.74	6	100°.84	9	136°.94	2	173°.03
1	29°.22	4	65°.32	7	101°.41	2.40	137°.51	3	173°.61
2	29°.79	1.15	65°.89	8	101°.99	1	138°.08	4	174°.18
3	30°.37	6	66°.46	9	102°.56	2	138°.66	3.05	174°.75
4	30°.94	7	67°.04	1.80	103°.13	3	139°.23	6	175°.33
.55	31°.51	8	67°.61	1	103°.71	4	139°.80	7	175°.90
6	32°.09	9	68°.18	2	104°.28	2.45	140°.37	8	176°.47
7	32°.66	1.20	68°.75	3	104°.85	6	140°.95	9	177°.04
8	33°.23	1	69°.33	4	105°.42	7	141°.52	3.10	177°.62
9	33°.80	2	69°.90	1.85	106°.00	8	142°.09	1	178°.19
.60	34°.38	3	70°.47	6	106°.57	9	142°.67	2	178°.76
1	34°.95	4	71°.05	7	107°.14	2.50	143°.24	3	179°.34
2	35°.52	1.25	71°.62	8	107°.72	1	143°.81	4	179°.91
3	36°.10	6	72°.19	9	108°.29	2	144°.39	3.15	180°.48

Interpolation	
.0002	0°.01
04	.02
06	.03
08	.05
.0010	0°.06
12	.07
14	.08
16	.09
18	.10
.0020	0°.11
22	.13
24	.14
26	.15
28	.16
.0030	0°.17
32	.18
34	.19
36	.21
38	.22
.0040	0°.23
42	.24
44	.25
46	.26
48	.28
.0050	0°.29
52	.30
54	.31
56	.32
58	.33
.0060	0°.34
62	.36
64	.37
66	.38
68	.39
.0070	0°.40
72	.41
74	.42
76	.44
78	.45
.0080	0°.46
82	.47
84	.48
86	.49
88	.50
.0090	0°.52
92	.53
94	.54
96	.55
98	.56

Multiples of π		
1	3.1416	180°
2	6.2832	360°
3	9.4248	540°
4	12.5664	720°
5	15.7080	900°
6	18.8496	1080°
7	21.9911	1260°
8	25.1327	1440°
9	28.2743	1620°
10	31.4159	1800°

NATURAL SINES AND COSINES

Natural Sines at intervals of 0°.1, or 6′. (For 10′ intervals, see pp. 52–56)

Deg.	°.0 =(0′)	°.1 (6′)	°.2 (12′)	°.3 (18′)	°.4 (24′)	°.5 (30′)	°.6 (36′)	°.7 (42′)	°.8 (48′)	°.9 (54′)			Avg. dif.
											0.0000	90°	
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	17
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	17
2	0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	0523	87	17
3	0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	0698	86	17
4	0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	0.0872	85	17
5	0.0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	1045	84	17
6	1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	1219	83	17
7	1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	1392	82	17
8	1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	1564	81	17
9	1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	0.1736	80°	17
10°	0.1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	1908	79	17
11	1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	2079	78	17
12	2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	2250	77	17
13	2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	2419	76	17
14	2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	0.2588	75	17
15	0.2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	2756	74	17
16	2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	2924	73	17
17	2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3090	72	17
18	3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3256	71	17
19	3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	0.3420	70°	16
20°	0.3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3584	69	16
21	3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3746	68	16
22	3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3907	67	16
23	3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	4067	66	16
24	4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	0.4226	65	16
25	0.4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	4384	64	16
26	4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	4540	63	16
27	4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	4695	62	16
28	4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	4848	61	15
29	4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	0.5000	60°	15
30°	0.5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	5150	59	15
31	5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	5299	58	15
32	5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	5446	57	15
33	5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	5592	56	15
34	5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	0.5736	55	14
35	0.5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	5878	54	14
36	5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	6018	53	14
37	6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	6157	52	14
38	6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	6293	51	14
39	6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	0.6428	50°	13
40°	0.6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	6561	49	13
41	6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	6691	48	13
42	6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	6820	47	13
43	6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	6947	46	13
44	6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	0.7071	45°	12
45°	0.7071												
		°.9 =(54′)	°.8 (48′)	°.7 (42′)	°.6 (36′)	°.5 (30′)	°.4 (24′)	°.3 (18′)	°.2 (12′)	°.1 (6′)	°.0 (0′)	Deg.	

(For graphs, see p. 174.) **Natural Cosines**

NATURAL SINES AND COSINES (*continued*)

Natural Sines at intervals of 0°.1, or 6′. (For 10′ intervals, see pp. 52–56)

Deg.	°.0 =(0′)	°.1 (6′)	°.2 (12′)	°.3 (18′)	°.4 (24′)	°.5 (30′)	°.6 (36′)	°.7 (42′)	°.8 (48′)	°.9 (54′)			Avg. diff.
											0.7071	45°	
45°	0.7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	7193	44	12
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	7314	43	12
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	7431	42	12
48	7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	7547	41	12
49	7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	0.7660	40°	11
50°	0.7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	7771	39	11
51	7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	7880	38	11
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	7986	37	11
53	7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	8090	36	10
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	0.8192	35	10
55	0.8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	8290	34	10
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	8387	33	10
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	8480	32	9
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	8572	31	9
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	0.8660	30°	9
60°	0.8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	8746	29	9
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	8829	28	8
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	8910	27	8
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	8988	26	8
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	0.9063	25	7
65	0.9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	9135	24	7
66	9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	9205	23	7
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	9272	22	7
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	9336	21	6
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	0.9397	20°	6
70°	0.9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	9455	19	6
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	9511	18	6
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	9563	17	5
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	9613	16	5
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	0.9659	15	5
75	0.9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	9703	14	4
76	9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	9744	13	4
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	9781	12	4
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	9816	11	3
79	9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	0.9848	10°	3
80°	0.9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	9877	9	3
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	9903	8	3
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	9925	7	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	9945	6	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0.9962	5	2
85	0.9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	9976	4	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	9986	3	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	9994	2	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0.9998	1	0
89	0.9998	9999	9999	9999	9999	0000	0000	0000	0000	0000	1.0000	0°	0
90°	1.0000												
		°.9 =(54′)	°.8 (48′)	°.7 (42′)	°.6 (36′)	°.5 (30′)	°.4 (24′)	°.3 (18′)	°.2 (12′)	°.1 (6′)	°.0 (0′)	Deg.	

Natural Cosines

NATURAL TANGENTS AND COTANGENTS

Natural Tangents at intervals of 0°.1, or 6′. (For 10′ intervals, see pp. 52–56)

Deg.	°.0 =(0′)	°.1 (6′)	°.2 (12′)	°.3 (18′)	°.4 (24′)	°.5 (30′)	°.6 (36′)	°.7 (42′)	°.8 (48′)	°.9 (54′)			Avg. diff.
											0.0000	90°	
0°	0.0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	0175	89	17
1	0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	0349	88	17
2	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	0524	87	17
3	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	0699	86	18
4	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	0.0875	85	18
5	0.0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	1051	84	18
6	1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	1228	83	18
7	1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	1405	82	18
8	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	1584	81	18
9	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	0.1763	80°	18
10°	0.1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	1944	79	18
11	1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	2126	78	18
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	2309	77	18
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	2493	76	18
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	0.2679	75	19
15	0.2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	2867	74	19
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3057	73	19
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3249	72	19
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3443	71	19
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	0.3640	70°	20
20°	0.3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3839	69	20
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	4040	68	20
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	4245	67	21
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	4452	66	21
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	0.4663	65	21
25	0.4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4877	64	21
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	5095	63	22
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	5317	62	22
28	5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	5543	61	23
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	0.5774	60°	23
30°	0.5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	6009	59	24
31	6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	6249	58	24
32	6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	6494	57	25
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	6745	56	25
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	0.7002	55	26
35	0.7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	7265	54	26
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	7536	53	27
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	7813	52	28
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	8098	51	28
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	0.8391	50°	29
40°	0.8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	8693	49	30
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	9004	48	31
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	9325	47	32
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	0.9657	46	33
44	0.9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	1.0000	45°	34
45°	1.0000												
		°.9 =(54′)	°.8 (48′)	°.7 (42′)	°.6 (36′)	°.5 (30′)	°.4 (24′)	°.3 (18′)	°.2 (12′)	°.1 (6′)	°.0 (0′)	Deg.	

(For graphs, see p. 174.)

Natural Cotangents

NATURAL TANGENTS AND COTANGENTS (*continued*)

Natural Tangents at intervals of 0°.1, or 6′. (For 10′ intervals, see pp. 52–56)

Deg.	°.0 =(0′)	°.1 (6′)	°.2 (12′)	°.3 (18′)	°.4 (24′)	°.5 (30′)	°.6 (36′)	°.7 (42′)	°.8 (48′)	°.9 (54′)			Avg. diff.
											1.0000	45°	
45°	1.0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	0355	44	35
46	0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	0724	43	37
47	0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	1106	42	38
48	1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	1504	41	40
49	1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	1.1918	40°	41
50°	1.1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	2349	39	43
51	2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	2799	38	45
52	2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	3270	37	47
53	3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	3764	36	49
54	3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	1.4281	35	52
55	1.4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	4826	34	55
56	4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	5399	33	57
57	5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	6003	32	60
58	6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	6643	31	64
59	1.6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	1.7321	30°	67
60°	1.732	1.739	1.746	1.753	1.760	1.767	1.775	1.782	1.789	1.797	1.804	29	7
61	1.804	1.811	1.819	1.827	1.834	1.842	1.849	1.857	1.865	1.873	1.881	28	8
62	1.881	1.889	1.897	1.905	1.913	1.921	1.929	1.937	1.946	1.954	1.963	27	8
63	1.963	1.971	1.980	1.988	1.997	2.006	2.014	2.023	2.032	2.041	2.050	26	9
64	2.050	2.059	2.069	2.078	2.087	2.097	2.106	2.116	2.125	2.135	2.145	25	9
65	2.145	2.154	2.164	2.174	2.184	2.194	2.204	2.215	2.225	2.236	2.246	24	10
66	2.246	2.257	2.267	2.278	2.289	2.300	2.311	2.322	2.333	2.344	2.356	23	11
67	2.356	2.367	2.379	2.391	2.402	2.414	2.426	2.438	2.450	2.463	2.475	22	12
68	2.475	2.488	2.500	2.513	2.526	2.539	2.552	2.565	2.578	2.592	2.605	21	13
69	2.605	2.619	2.633	2.646	2.660	2.675	2.689	2.703	2.718	2.733	2.747	20°	14
70°	2.747	2.762	2.778	2.793	2.808	2.824	2.840	2.856	2.872	2.888	2.904	19	16
71	2.904	2.921	2.937	2.954	2.971	2.989	3.006	3.024	3.042	3.060	3.078	18	17
72	3.078	3.096	3.115	3.133	3.152	3.172	3.191	3.211	3.230	3.251	3.271	17	19
73	3.271	3.291	3.312	3.333	3.354	3.376	3.398	3.420	3.442	3.465	3.487	16	22
74	3.487	3.511	3.534	3.558	3.582	3.606	3.630	3.655	3.681	3.706	3.732	15	24
75	3.732	3.758	3.785	3.812	3.839	3.867	3.895	3.923	3.952	3.981	4.011	14	28
76	4.011	4.041	4.071	4.102	4.134	4.165	4.198	4.230	4.264	4.297	4.331	13	32
77	4.331	4.366	4.402	4.437	4.474	4.511	4.548	4.586	4.625	4.665	4.705	12	37
78	4.705	4.745	4.787	4.829	4.872	4.915	4.959	5.005	5.050	5.097	5.145	11	44
79	5.145	5.193	5.242	5.292	5.343	5.396	5.449	5.503	5.558	5.614	5.671	10°	53
80°	5.671	5.730	5.789	5.850	5.912	5.976	6.041	6.107	6.174	6.243	6.314	9	
81	6.314	6.386	6.460	6.535	6.612	6.691	6.772	6.855	6.940	7.026	7.115	8	
82	7.115	7.207	7.300	7.396	7.495	7.596	7.700	7.806	7.916	8.028	8.144	7	
83	8.144	8.264	8.386	8.513	8.643	8.777	8.915	9.058	9.205	9.357	9.514	6	
84	9.514	9.677	9.845	10.02	10.20	10.39	10.58	10.78	10.99	11.20	11.43	5	
85	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95	14.30	4	
86	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46	19.08	3	
87	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27	28.64	2	
88	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08	57.29	1	
89	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0	∞	0°	
90°	∞												
		°.9 =(54′)	°.8 (48′)	°.7 (42′)	°.6 (36′)	°.5 (30′)	°.4 (24′)	°.3 (18′)	°.2 (12′)	°.1 (6′)	°.0 (0′)	Deg.	

Natural Cotangents

NATURAL SECANTS AND COSECANTS

Natural Secants at intervals of 0°.1, or 6′. (For 10′ intervals, see pp. 52–56)

Deg.	°.0 =(0′)	°.1 (6′)	°.2 (12′)	°.3 (18′)	°.4 (24′)	°.5 (30′)	°.6 (36′)	°.7 (42′)	°.8 (48′)	°.9 (54′)			Avg. diff.
											1.0000	**90°**	
0°	1.0000	0000	0000	0000	0000	0000	0001	0001	0001	0001	0002	89	0
1	0002	0002	0002	0003	0003	0003	0004	0004	0005	0006	0006	88	0
2	0006	0007	0007	0008	0009	0010	0010	0011	0012	0013	0014	87	1
3	0014	0015	0016	0017	0018	0019	0020	0021	0022	0023	0024	86	1
4	0024	0026	0027	0028	0030	0031	0032	0034	0035	0037	1.0038	85	1
5	1.0038	0040	0041	0043	0045	0046	0048	0050	0051	0053	0055	84	2
6	0055	0057	0059	0061	0063	0065	0067	0069	0071	0073	0075	83	2
7	0075	0077	0079	0082	0084	0086	0089	0091	0093	0096	0098	82	2
8	0098	0101	0103	0106	0108	0111	0114	0116	0119	0122	0125	81	3
9	0125	0127	0130	0133	0136	0139	0142	0145	0148	0151	1.0154	**80°**	3
10°	1.0154	0157	0161	0164	0167	0170	0174	0177	0180	0184	0187	79	3
11	0187	0191	0194	0198	0201	0205	0209	0212	0216	0220	0223	78	4
12	0223	0227	0231	0235	0239	0243	0247	0251	0255	0259	0263	77	4
13	0263	0267	0271	0276	0280	0284	0288	0293	0297	0302	0306	76	4
14	0306	0311	0315	0320	0324	0329	0334	0338	0343	0348	1.0353	75	5
15	1.0353	0358	0363	0367	0372	0377	0382	0388	0393	0398	0403	74	5
16	0403	0408	0413	0419	0424	0429	0435	0440	0446	0451	0457	73	5
17	0457	0463	0468	0474	0480	0485	0491	0497	0503	0509	0515	72	6
18	0515	0521	0527	0533	0539	0545	0551	0557	0564	0570	0576	71	6
19	0576	0583	0589	0595	0602	0608	0615	0622	0628	0635	1.0642	**70°**	7
20°	1.0642	0649	0655	0662	0669	0676	0683	0690	0697	0704	0711	69	7
21	0711	0719	0726	0733	0740	0748	0755	0763	0770	0778	0785	68	7
22	0785	0793	0801	0808	0816	0824	0832	0840	0848	0856	0864	67	8
23	0864	0872	0880	0888	0896	0904	0913	0921	0929	0938	0946	66	8
24	0946	0955	0963	0972	0981	0989	0998	1007	1016	1025	1.1034	65	9
25	1.1034	1043	1052	1061	1070	1079	1089	1098	1107	1117	1126	64	9
26	1126	1136	1145	1155	1164	1174	1184	1194	1203	1213	1223	63	10
27	1223	1233	1243	1253	1264	1274	1284	1294	1305	1315	1326	62	10
28	1326	1336	1347	1357	1368	1379	1390	1401	1412	1423	1434	61	11
29	1434	1445	1456	1467	1478	1490	1501	1512	1524	1535	1.1547	**60°**	11
30°	1.1547	1559	1570	1582	1594	1606	1618	1630	1642	1654	1666	59	12
31	1666	1679	1691	1703	1716	1728	1741	1753	1766	1779	1792	58	13
32	1792	1805	1818	1831	1844	1857	1870	1883	1897	1910	1924	57	13
33	1924	1937	1951	1964	1978	1992	2006	2020	2034	2048	2062	56	14
34	2062	2076	2091	2105	2120	2134	2149	2163	2178	2193	1.2208	55	15
35	1.2208	2223	2238	2253	2268	2283	2299	2314	2329	2345	2361	54	15
36	2361	2376	2392	2408	2424	2440	2456	2472	2489	2505	2521	53	16
37	2521	2538	2554	2571	2588	2605	2622	2639	2656	2673	2690	52	17
38	2690	2708	2725	2742	2760	2778	2796	2813	2831	2849	2868	51	18
39	2868	2886	2904	2923	2941	2960	2978	2997	3016	3035	1.3054	**50°**	19
40°	1.3054	3073	3093	3112	3131	3151	3171	3190	3210	3230	3250	49	20
41	3250	3270	3291	3311	3331	3352	3373	3393	3414	3435	3456	48	21
42	3456	3478	3499	3520	3542	3563	3585	3607	3629	3651	3673	47	22
43	3673	3696	3718	3741	3763	3786	3809	3832	3855	3878	3902	46	23
44	3902	3925	3949	3972	3996	4020	4044	4069	4093	4118	1.4142	**45°**	24
45°	1.4142												
		°.9 =(54′)	°.8 (48′)	°.7 (42′)	°.6 (36′)	°.5 (30′)	°.4 (24′)	°.3 (18′)	°.2 (12′)	°.1 (6′)	°.0 (0′)	Deg.	

(For graphs, see p. 174.)

Natural Cosecants

NATURAL SECANTS AND COSECANTS *(continued)*

Natural Secants at intervals of 0°.1, or 6′. (For 10′ intervals, see pp. 52–56)

Deg.	°.0 =(0′)	°.1 (6′)	°.2 (12′)	°.3 (18′)	°.4 (24′)	°.5 (30′)	°.6 (36′)	°.7 (42′)	°.8 (48′)	°.9 (54′)			Avg. diff.
											1.4142	45°	
45°	1.4142	4167	4192	4217	4242	4267	4293	4318	4344	4370	4396	44	25
46	4396	4422	4448	4474	4501	4527	4554	4581	4608	4635	4663	43	27
47	4663	4690	4718	4746	4774	4802	4830	4859	4887	4916	4945	42	28
48	4945	4974	5003	5032	5062	5092	5121	5151	5182	5212	5243	41	30
49	5243	5273	5304	5335	5366	5398	5429	5461	5493	5525	1.5557	40°	31
50°	1.5557	5590	5622	5655	5688	5721	5755	5788	5822	5856	5890	39	33
51	5890	5925	5959	5994	6029	6064	6099	6135	6171	6207	6243	38	35
52	6243	6279	6316	6353	6390	6427	6464	6502	6540	6578	6616	37	37
53	6616	6655	6694	6733	6772	6812	6852	6892	6932	6972	7013	36	40
54	7013	7054	7095	7137	7179	7221	7263	7305	7348	7391	1.7434	35	42
55	1.7434	7478	7522	7566	7610	7655	7700	7745	7791	7837	7883	34	45
56	7883	7929	7976	8023	8070	8118	8166	8214	8263	8312	8361	33	48
57	8361	8410	8460	8510	8561	8612	8663	8714	8766	8818	8871	32	51
58	8871	8924	8977	9031	9084	9139	9194	9249	9304	9360	1.9416	31	54
59	1.9416	9473	9530	9587	9645	9703	9762	9821	9880	9940	2.0000	30°	58
60°	2.000	2.006	2.012	2.018	2.025	2.031	2.037	2.043	2.050	2.056	2.063	29	6
61	2.063	2.069	2.076	2.082	2.089	2.096	2.103	2.109	2.116	2.123	2.130	28	7
62	2.130	2.137	2.144	2.151	2.158	2.166	2.173	2.180	2.188	2.195	2.203	27	7
63	2.203	2.210	2.218	2.226	2.233	2.241	2.249	2.257	2.265	2.273	2.281	26	8
64	2.281	2.289	2.298	2.306	2.314	2.323	2.331	2.340	2.349	2.357	2.366	25	8
65	2.366	2.375	2.384	2.393	2.402	2.411	2.421	2.430	2.439	2.449	2.459	24	9
66	2.459	2.468	2.478	2.488	2.498	2.508	2.518	2.528	2.538	2.549	2.559	23	10
67	2.559	2.570	2.581	2.591	2.602	2.613	2.624	2.635	2.647	2.658	2.669	22	11
68	2.669	2.681	2.693	2.705	2.716	2.729	2.741	2.753	2.765	2.778	2.790	21	12
69	2.790	2.803	2.816	2.829	2.842	2.855	2.869	2.882	2.896	2.910	2.924	20°	13
70°	2.924	2.938	2.952	2.967	2.981	2.996	3.011	3.026	3.041	3.056	3.072	19	15
71	3.072	3.087	3.103	3.119	3.135	3.152	3.168	3.185	3.202	3.219	3.236	18	16
72	3.236	3.254	3.271	3.289	3.307	3.326	3.344	3.363	3.382	3.401	3.420	17	18
73	3.420	3.440	3.460	3.480	3.500	3.521	3.542	3.563	3.584	3.606	3.628	16	21
74	3.628	3.650	3.673	3.695	3.719	3.742	3.766	3.790	3.814	3.839	3.864	15	24
75	3.864	3.889	3.915	3.941	3.967	3.994	4.021	4.049	4.077	4.105	4.134	14	27
76	4.134	4.163	4.192	4.222	4.253	4.284	4.315	4.347	4.379	4.412	4.445	13	31
77	4.445	4.479	4.514	4.549	4.584	4.620	4.657	4.694	4.732	4.771	4.810	12	36
78	4.810	4.850	4.890	4.931	4.973	5.016	5.059	5.103	5.148	5.194	5.241	11	43
79	5.241	5.288	5.337	5.386	5.436	5.487	5.540	5.593	5.647	5.702	5.759	10°	52
80°	5.759	5.816	5.875	5.935	5.996	6.059	6.123	6.188	6.255	6.323	6.392	9	
81	6.392	6.464	6.537	6.611	6.687	6.765	6.845	6.927	7.011	7.097	7.185	8	
82	7.185	7.276	7.368	7.463	7.561	7.661	7.764	7.870	7.979	8.091	8.206	7	
83	8.206	8.324	8.446	8.571	8.700	8.834	8.971	9.113	9.259	9.411	9.567	6	
84	9.567	9.728	9.895	10.07	10.25	10.43	10.63	10.83	11.03	11.25	11.47	5	
85	11.47	11.71	11.95	12.20	12.47	12.75	13.03	13.34	13.65	13.99	14.34	4	
86	14.34	14.70	15.09	15.50	15.93	16.38	16.86	17.37	17.91	18.49	19.11	3	
87	19.11	19.77	20.47	21.23	22.04	22.93	23.88	24.92	26.05	27.29	28.65	2	
88	28.65	30.16	31.84	33.71	35.81	38.20	40.93	44.08	47.75	52.09	57.30	1	
89	57.30	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0	∞	0°	
90°	∞												
		°.9 =(54′)	°.8 (48′)	°.7 (42′)	°.6 (36′)	°.5 (30′)	°.4 (24′)	°.3 (18′)	°.2 (12′)	°.1 (6′)	°.0 (0′)	Deg.	

Natural Cosecants

TRIGONOMETRIC FUNCTIONS (at intervals of 10′)

Annex −10 in columns marked *. (For 0.°1 intervals, see pp. 46–51)

Degrees	Radians	Sines Nat.	Sines Log.*	Cosines Nat.	Cosines Log.*	Tangents Nat.	Tangents Log.*	Cotangents Nat.	Cotangents Log.		
0° 00′	0.0000	.0000	∞	1.0000	0.0000	.0000	∞	∞	∞	1.5708	90° 00′
10	0.0029	.0029	7.4637	1.0000	.0000	.0029	7.4637	343.77	2.5363	1.5679	50
20	0 0058	.0058	.7648	1.0000	.0000	.0058	.7648	171.89	.2352	1.5650	40
30	0 0087	.0087	.9408	1.0000	.0000	.0087	.9409	114.59	.0591	1.5621	30
40	0.0116	.0116	8.0658	0.9999	.0000	.0116	8.0658	85.940	1.9342	1.5592	20
50	0.0145	.0145	.1627	.9999	.0000	.0145	.1627	68.750	.8373	1.5563	10
1° 00′	0.0175	.0175	8.2419	.9998	9.9999	.0175	8.2419	57.290	1.7581	1.5533	89° 00′
10	0.0204	.0204	.3088	.9998	.9999	.0204	.3089	49.104	.6911	1.5504	50
20	0.0233	.0233	.3668	.9997	.9999	.0233	.3669	42.964	.6331	1.5475	40
30	0.0262	.0262	.4179	.9997	.9999	.0262	.4181	38.188	.5819	1.5446	30
40	0.0291	.0291	.4637	.9996	.9998	.0291	.4638	34.368	.5362	1.5417	20
50	0.0320	.0320	.5050	.9995	.9998	.0320	.5053	31.242	.4947	1.5388	10
2° 00′	0.0349	.0349	8.5428	.9994	9.9997	.0349	8.5431	28.636	1.4569	1.5359	88° 00′
10	0.0378	.0378	.5776	.9993	.9997	.0378	.5779	26.432	.4221	1.5330	50
20	0.0407	.0407	.6097	.9992	.9996	.0407	.6101	24.542	.3899	1.5301	40
30	0.0436	.0436	.6397	.9990	.9996	.0437	.6401	22.904	.3599	1.5272	30
40	0.0465	.0465	.6677	.9989	.9995	.0466	.6682	21.470	.3318	1.5243	20
50	0.0495	.0494	.6940	.9988	.9995	.0495	.6945	20.206	.3055	1.5213	10
3° 00′	0.0524	.0523	8.7188	.9986	9.9994	.0524	8.7194	19.081	1.2806	1.5184	87° 00′
10	0.0553	.0552	.7423	.9985	.9993	.0553	.7429	18.075	.2571	1.5155	50
20	0.0582	.0581	.7645	.9983	.9993	.0582	.7652	17.169	.2348	1.5126	40
30	0.0611	.0610	.7857	.9981	.9992	.0612	.7865	16.350	.2135	1.5097	30
40	0.0640	.0640	.8059	.9980	.9991	.0641	.8067	15.605	.1933	1.5068	20
50	0.0669	.0669	.8251	.9978	.9990	.0670	.8261	14.924	.1739	1.5039	10
4° 00′	0.0698	.0698	8.8436	.9976	9.9989	.0699	8.8446	14.301	1.1554	1.5010	86° 00
10	0.0727	.0727	.8613	.9974	.9989	.0729	.8624	13.727	.1376	1.4981	50
20	0.0756	.0756	.8783	.9971	.9988	.0758	.8795	13.197	.1205	1.4952	40
30	0.0785	.0785	.8946	.9969	.9987	.0787	.8960	12.706	.1040	1.4923	30
40	0.0814	.0814	.9104	.9967	.9986	.0816	.9118	12.251	.0882	1.4893	20
50	0.0844	.0843	.9256	.9964	.9985	.0846	.9272	11.826	.0728	1.4864	10
5° 00′	0.0873	.0872	8.9403	.9962	9.9983	.0875	8.9420	11.430	1.0580	1.4835	85° 00′
10	0.0902	.0901	.9545	.9959	.9982	.0904	.9563	11.059	.0437	1.4806	50
20	0.0931	.0929	.9682	.9957	.9981	.0934	.9701	10.712	.0299	1.4777	40
30	0.0960	.0958	.9816	.9954	.9980	.0963	.9836	10.385	.0164	1.4748	30
40	0.0989	.0987	.9945	.9951	.9979	.0992	.9966	10.078	.0034	1.4719	20
50	0.1018	.1016	9.0070	.9948	.9977	.1022	9.0093	9.7882	0.9907	1.4690	10
6° 00′	0.1047	.1045	9.0192	.9945	9.9976	.1051	9.0216	9.5144	0.9784	1.4661	84° 00′
10	0.1076	.1074	.0311	.9942	.9975	.1080	.0336	9.2553	.9664	1.4632	50
20	0.1105	.1103	.0426	.9939	.9973	.1110	.0453	9.0098	.9547	1.4603	40
30	0.1134	.1132	.0539	.9936	.9972	.1139	.0567	8.7769	.9433	1.4574	30
40	0.1164	.1161	.0648	.9932	.9971	.1169	.0678	8.5555	.9322	1.4544	20
50	0.1193	.1190	.0755	.9929	.9969	.1198	.0786	8.3450	.9214	1.4515	10
7° 00′	0.1222	.1219	9.0859	.9925	9.9968	.1228	9.0891	8.1443	0.9109	1.4486	83° 00′
10	0.1251	.1248	.0961	.9922	.9966	.1257	.0995	7.9530	.9005	1.4457	50
20	0.1280	.1276	.1060	.9918	.9964	.1287	.1096	7.7704	.8904	1.4428	40
30	0.1309	.1305	.1157	.9914	.9963	.1317	.1194	7.5958	.8806	1.4399	30
40	0.1338	.1334	.1252	.9911	.9961	.1346	.1291	7.4287	.8709	1.4370	20
50	0.1367	.1363	.1345	.9907	.9959	.1376	.1385	7.2687	.8615	1.4341	10
8° 00′	0.1396	.1392	9.1436	.9903	9.9958	.1405	9.1478	7.1154	0.8522	1.4312	82° 00′
10	0.1425	.1421	.1525	.9899	.9956	.1435	.1569	6.9682	.8431	1.4283	50
20	0.1454	.1449	.1612	.9894	.9954	.1465	.1658	6.8269	.8342	1.4254	40
30	0.1484	.1478	.1697	.9890	.9952	.1495	.1745	6.6912	.8255	1.4224	30
40	0.1513	.1507	.1781	.9886	.9950	.1524	.1831	6.5606	.8169	1.4195	20
50	0.1542	.1536	.1863	.9881	.9948	.1554	.1915	6.4348	.8085	1.4166	10
9° 00′	0.1571	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	1.4137	81° 00′
		Cosines Nat.	Cosines Log.*	Sines Nat.	Sines Log.*	Cotangents Nat.	Cotangents Log.*	Tangents Nat.	Tangents Log.	Radians	Degrees

TRIGONOMETRIC FUNCTIONS (*continued*)

Annex −10 in columns marked *. (For 0.°1 intervals, see pp. 46–51)

Degrees	Radians	Sines Nat.	Sines Log.*	Cosines Nat.	Cosines Log.*	Tangents Nat.	Tangents Log.*	Cotangents Nat.	Cotangents Log.		
9° 00′	0.1571	.1564	9.1943	.9877	9.9946	.1584	9.1997	6.3138	0.8003	1.4137	81° 00′
10	0.1600	.1593	.2022	.9872	.9944	.1614	.2078	6.1970	.7922	1.4108	50
20	0.1629	.1622	.2100	.9868	.9942	.1644	.2158	6.0844	.7842	1.4079	40
30	0.1658	.1650	.2176	.9863	.9940	.1673	.2236	5.9758	.7764	1.4050	30
40	0.1687	.1679	.2251	.9858	.9938	.1703	.2313	5.8708	.7687	1.4021	20
50	0.1716	.1708	.2324	.9853	.9936	.1733	.2389	5.7694	.7611	1.3992	10
10° 00′	0.1745	.1736	9.2397	.9848	9.9934	.1763	9.2463	5.6713	0.7537	1.3963	80° 00′
10	0.1774	.1765	.2468	.9843	.9931	.1793	.2536	5.5764	.7464	1.3934	50
20	0.1804	.1794	.2538	.9838	.9929	.1823	.2609	5.4845	.7391	1.3904	40
30	0.1833	.1822	.2606	.9833	.9927	.1853	.2680	5.3955	.7320	1.3875	30
40	0.1862	.1851	.2674	.9827	.9924	.1883	.2750	5.3093	.7250	1.3846	20
50	0.1891	.1880	.2740	.9822	.9922	.1914	.2819	5.2257	.7181	1.3817	10
11° 00′	0.1920	.1908	9.2806	.9816	9.9919	.1944	9.2887	5.1446	0.7113	1.3788	79° 00′
10	0.1949	.1937	.2870	.9811	.9917	.1974	.2953	5.0658	.7047	1.3759	50
20	0.1978	.1965	.2934	.9805	.9914	.2004	.3020	4.9894	.6980	1.3730	40
30	0.2007	.1994	.2997	.9799	.9912	.2035	.3085	4.9152	.6915	1.3701	30
40	0.2036	.2022	.3058	.9793	.9909	.2065	.3149	4.8430	.6851	1.3672	20
50	0.2065	.2051	.3119	.9787	.9907	.2095	.3212	4.7729	.6788	1.3643	10
12° 00′	0.2094	.2079	9.3179	.9781	9.9904	.2126	9.3275	4.7046	0.6725	1.3614	78° 00′
10	0.2123	.2108	.3238	.9775	.9901	.2156	.3336	4.6382	.6664	1.3584	50
20	0.2153	.2136	.3296	.9769	.9899	.2186	.3397	4.5736	.6603	1.3555	40
30	0.2182	.2164	.3353	.9763	.9896	.2217	.3458	4.5107	.6542	1.3526	30
40	0.2211	.2193	.3410	.9757	.9893	.2247	.3517	4.4494	.6483	1.3497	20
50	0.2240	.2221	.3466	.9750	.9890	.2278	.3576	4.3897	.6424	1.3468	10
13° 00′	0.2269	.2250	9.3521	.9744	9.9887	.2309	9.3634	4.3315	0.6366	1.3439	77° 00′
10	0.2298	.2278	.3575	.9737	.9884	.2339	.3691	4.2747	.6309	1.3410	50
20	0.2327	.2306	.3629	.9730	.9881	.2370	.3748	4.2193	.6252	1.3381	40
30	0.2356	.2334	.3682	.9724	.9878	.2401	.3804	4.1653	.6196	1.3352	30
40	0.2385	.2363	.3734	.9717	.9875	.2432	.3859	4.1126	.6141	1.3323	20
50	0.2414	.2391	.3786	.9710	.9872	.2462	.3914	4.0611	.6086	1.3294	10
14° 00′	0.2443	.2419	9.3837	.9703	9.9869	.2493	9.3968	4.0108	0.6032	1.3265	76° 00′
10	0.2473	.2447	.3887	.9696	.9866	.2524	.4021	3.9617	.5979	1.3235	50
20	0.2502	.2476	.3937	.9689	.9863	.2555	.4074	3.9136	.5926	1.3206	40
30	0.2531	.2504	.3986	.9681	.9859	.2586	.4127	3.8667	.5873	1.3177	30
40	0.2560	.2532	.4035	.9674	.9856	.2617	.4178	3.8208	.5822	1.3148	20
50	0.2589	.2560	.4083	.9667	.9853	.2648	.4230	3.7760	.5770	1.3119	10
15° 00′	0.2618	.2588	9.4130	.9659	9.9849	.2679	9.4281	3.7321	0.5719	1.3090	75° 00′
10	0.2647	.2616	.4177	.9652	.9846	.2711	.4331	3.6891	.5669	1.3061	50
20	0.2676	.2644	.4223	.9644	.9843	.2742	.4381	3.6470	.5619	1.3032	40
30	0.2705	.2672	.4269	.9636	.9839	.2773	.4430	3.6059	.5570	1.3003	30
40	0.2734	.2700	.4314	.9628	.9836	.2805	.4479	3.5656	.5521	1.2974	20
50	0.2763	.2728	.4359	.9621	.9832	.2836	.4527	3.5261	.5473	1.2945	10
16° 00′	0.2793	.2756	9.4403	.9613	9.9828	.2867	9.4575	3.4874	0.5425	1.2915	74° 00′
10	0.2822	.2784	.4447	.9605	.9825	.2899	.4622	3.4495	.5378	1.2886	50
20	0.2851	.2812	.4491	.9596	.9821	.2931	.4669	3.4124	.5331	1.2857	40
30	0.2880	.2840	.4533	.9588	.9817	.2962	.4716	3.3759	.5284	1.2828	30
40	0.2909	.2868	.4576	.9580	.9814	.2994	.4762	3.3402	.5238	1.2799	20
50	0.2938	.2896	.4618	.9572	.9810	.3026	.4808	3.3052	.5192	1.2770	10
17° 00′	0.2967	.2924	9.4659	.9563	9.9806	.3057	9.4853	3.2709	0.5147	1.2741	73° 00′
10	0.2996	.2952	.4700	.9555	.9802	.3089	.4898	3.2371	.5102	1.2712	50
20	0.3025	.2979	.4741	.9546	.9798	.3121	.4943	3.2041	.5057	1.2683	40
30	0.3054	.3007	.4781	.9537	.9794	.3153	.4987	3.1716	.5013	1.2654	30
40	0.3083	.3035	.4821	.9528	.9790	.3185	.5031	3.1397	.4969	1.2625	20
50	0.3113	.3062	.4861	.9520	.9786	.3217	.5075	3.1084	.4925	1.2595	10
18° 00′	0.3142	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	1.2566	72° 00′
		Nat.	Log.*	Nat.	Log.*	Nat.	Log.*	Nat.	Log.		
		Cosines		Sines		Cotangents		Tangents		Radians	Degrees

TRIGONOMETRIC FUNCTIONS (*continued*)

Annex −10 in columns marked *. (For 0.°1 intervals, see pp. 46–51)

Degrees	Radians	Sines Nat.	Sines Log.*	Cosines Nat.	Cosines Log.*	Tangents Nat.	Tangents Log.*	Cotangents Nat.	Cotangents Log.		
18° 00′	0.3142	.3090	9.4900	.9511	9.9782	.3249	9.5118	3.0777	0.4882	1.2566	72° 00′
10	0.3171	.3118	.4939	.9502	.9778	.3281	.5161	3.0475	.4839	1.2537	50
20	0.3200	.3145	.4977	.9492	.9774	.3314	.5203	3.0178	.4797	1.2508	40
30	0.3229	.3173	.5015	.9483	.9770	.3346	.5245	2.9887	.4755	1.2479	30
40	0.3258	.3201	.5052	.9474	.9765	.3378	.5287	2.9600	.4713	1.2450	20
50	0.3287	.3228	.5090	.9465	.9761	.3411	.5329	2.9319	.4671	1.2421	10
19° 00′	0.3316	.3256	9.5126	.9455	9.9757	.3443	9.5370	2.9042	0.4630	1.2392	71° 00′
10	0.3345	.3283	.5163	.9446	.9752	.3476	.5411	2.8770	.4589	1.2363	50
20	0.3374	.3311	.5199	.9436	.9748	.3508	.5451	2.8502	.4549	1.2334	40
30	0.3403	.3338	.5235	.9426	.9743	.3541	.5491	2.8239	.4509	1.2305	30
40	0.3432	.3365	.5270	.9417	.9739	.3574	.5531	2.7980	.4469	1.2275	20
50	0.3462	.3393	.5306	.9407	.9734	.3607	.5571	2.7725	.4429	1 2246	10
20° 00′	0.3491	.3420	9.5341	.9397	9.9730	.3640	9.5611	2.7475	0.4389	1.2217	70° 00′
10	0.3520	.3448	.5375	.9387	.9725	.3673	.5650	2.7228	.4350	1.2188	50
20	0.3549	3475	.5409	.9377	.9721	.3706	.5689	2.6985	.4311	1.2159	40
30	0.3578	.3502	.5443	.9367	.9716	.3739	.5727	2.6746	.4273	1.2130	30
40	0.3607	.3529	.5477	.9356	.9711	.3772	.5766	2.6511	.4234	1.2101	20
50	0.3636	.3557	.5510	.9346	.9706	.3805	.5804	2.6279	.4196	1.2072	10
21° 00′	0.3665	.3584	9.5543	.9336	9.9702	.3839	9.5842	2.6051	0.4158	1.2043	69° 00′
10	0.3694	.3611	.5576	.9325	.9697	.3872	.5879	2.5826	.4121	1.2014	50
20	0.3723	.3638	.5609	.9315	.9692	.3906	.5917	2.5605	.4083	1.1985	40
30	0.3752	.3665	.5641	.9304	.9687	.3939	.5954	2.5386	.4046	1.1956	30
40	0.3782	.3692	.5673	.9293	.9682	.3973	.5991	2.5172	.4009	1.1926	20
50	0.3811	.3719	.5704	.9283	.9677	.4006	.6028	2.4960	.3972	1.1897	10
22° 00′	0.3840	.3746	9.5736	.9272	9.9672	.4040	9.6064	2.4751	0.3936	1.1868	68° 00′
10	0.3869	.3773	.5767	.9261	.9667	.4074	.6100	2.4545	.3900	1.1839	50
20	0.3898	.3800	.5798	.9250	.9661	.4108	.6136	2.4342	.3864	1.1810	40
30	0.3927	.3827	.5828	.9239	.9656	.4142	.6172	2.4142	.3828	1.1781	30
40	0.3956	.3854	.5859	.9228	.9651	.4176	.6208	2.3945	.3792	1.1752	20
50	0.3985	.3881	.5889	.9216	.9646	.4210	.6243	2.3750	.3757	1.1723	10
23° 00′	0.4014	.3907	9.5919	.9205	9.9640	.4245	9.6279	2.3559	0.3721	1.1694	67° 00′
10	0.4043	.3934	.5948	.9194	.9635	.4279	.6314	2.3369	.3686	1.1665	50
20	0.4072	.3961	.5978	.9182	.9629	.4314	.6348	2.3183	.3652	1.1636	40
30	0.4102	.3987	.6007	.9171	.9624	.4348	.6383	2.2998	.3617	1.1606	30
30	0.4131	.4014	.6036	.9159	.9618	.4383	.6417	2.2817	.3583	1.1577	20
50	0.4160	.4041	.6065	.9147	.9613	.4417	.6452	2.2637	.3548	1.1548	10
24° 00′	0.4189	.4067	9.6093	.9135	9.9607	.4452	9.6486	2.2460	0.3514	1.1519	66° 00′
10	0.4218	.4094	.6121	.9124	.9602	.4487	.6520	2.2286	.3480	1.1490	50
20	0.4247	.4120	.6149	.9112	.9596	.4522	.6553	2.2113	.3447	1.1461	40
30	0.4276	.4147	.6177	.9100	.9590	.4557	.6587	2.1943	.3413	1.1432	30
40	0.4305	.4173	.6205	.9088	.9584	.4592	.6620	2.1775	.3380	1.1403	20
50	0.4334	.4100	.6232	.9075	.9579	.4628	.6654	2.1609	.3346	1.1374	10
25° 00′	0.4363	.4226	9.6259	.9063	9.9573	.4663	9.6687	2.1445	0.3313	1.1345	65° 00′
10	0.4392	.4253	.6286	.9051	.9567	.4699	.6720	2.1283	.3280	1.1316	50
20	0.4422	.4279	.6313	.9038	.9561	.4734	.6752	2.1123	.3248	1.1286	40
30	0.4451	.4305	.6340	.9026	.9555	.4770	.6785	2.0965	.3215	1.1257	30
40	0.4480	.4331	.6366	.9013	.9549	.4806	.6817	2.0809	.3183	1.1228	20
50	0.4509	.4358	.6392	.9001	.9543	.4841	.6850	2.0655	.3150	1.1199	10
26° 00′	0.4538	.4384	9.6418	.8988	9.9537	.4877	9.6882	2.0503	0.3118	1.1170	64° 00′
10	0.4567	.4410	.6444	.8975	.9530	.4913	.6914	2.0353	.3086	1.1141	50
20	0.4596	.4436	.6470	.8962	.9524	.4950	.6946	2.0204	.3054	1.1112	40
30	0.4625	.4462	.6495	.8949	.9518	.4986	.6977	2.0057	.3023	1.1083	30
40	0.4654	.4488	.6521	.8936	.9512	.5022	.7009	1.9912	.2991	1.1054	20
50	0.4683	.4514	.6546	.8923	.9505	.5059	.7040	1.9768	.2960	1.1025	10
27° 00′	0.4712	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	1.0996	63° 00′
		Nat.	Log.*	Nat.	Log.*	Nat.	Log.*	Nat.	Log.		
		Cosines		Sines		Cotangents		Tangents		Radians	Degrees

TRIGONOMETRIC FUNCTIONS *(continued)*

Annex −10 in columns marked*. (For 0°.1 intervals, see pp. 46–51)

Degrees	Radians	Sines		Cosines		Tangents		Cotangents			
		Nat.	Log.*	Nat.	Log.*	Nat.	Log.*	Nat.	Log.		
27° 00′	0.4712	.4540	9.6570	.8910	9.9499	.5095	9.7072	1.9626	0.2928	1.0996	63° 00′
10	0.4741	.4566	.6595	.8897	.9492	.5132	.7103	1.9486	.2897	1.0966	50
20	0.4771	.4592	.6620	.8884	.9486	.5169	.7134	1.9347	.2866	1.0937	40
30	0.4800	.4617	.6644	.8870	.9479	.5206	.7165	1.9210	.2835	1.0908	30
40	0.4829	.4643	.6668	.8857	.9473	.5243	.7196	1.9074	.2804	1.0879	20
50	0.4858	.4669	.6692	.8843	.9466	.5280	.7226	1.8940	.2774	1.0850	10
28° 00′	0.4887	.4695	9.6716	.8829	9.9459	.5317	9.7257	1.8807	0.2743	1.0821	62° 00′
10	0.4916	.4720	.6740	.8816	.9453	.5354	.7287	1.8676	.2713	1.0792	50
20	0.4945	.4746	.6763	.8802	.9446	.5392	.7317	1.8546	.2683	1.0763	40
30	0.4974	.4772	.6787	.8788	.9439	.5430	.7348	1.8418	.2652	1.0734	30
40	0.5003	.4797	.6810	.8774	.9432	.5467	.7378	1.8291	.2622	1.0705	20
50	0.5032	.4823	.6833	.8760	.9425	.5505	.7408	1.8165	.2592	1.0676	10
29° 00′	0.5061	.4848	9.6856	.8746	9.9418	.5543	9.7438	1.8040	0.2562	1.0647	61° 00′
10	0.5091	.4874	.6878	.8732	.9411	.5581	.7467	1.7917	.2533	1.0617	50
20	0.5120	.4899	.6901	.8718	.9404	.5619	.7497	1.7796	.2503	1.0588	40
30	0.5149	.4924	.6923	.8704	.9397	.5658	.7526	1.7675	.2474	1.0559	30
40	0.5178	.4950	.6946	.8689	.9390	.5696	.7556	1.7556	.2444	1.0530	20
50	0.5207	.4975	.6968	.8675	.9383	.5735	.7585	1.7437	.2415	1.0501	10
30° 00′	0.5236	.5000	9.6990	.8660	9.9375	.5774	9.7614	1.7321	0.2386	1.0472	60° 00′
10	0.5265	.5025	.7012	.8646	.9368	.5812	.7644	1.7205	.2356	1.0443	50
20	0.5294	.5050	.7033	.8631	.9361	.5851	.7673	1.7090	.2327	1.0414	40
30	0.5323	.5075	.7055	.8616	.9353	.5890	.7701	1.6977	.2299	1.0385	30
40	0.5352	.5100	.7076	.8601	.9346	.5930	.7730	1.6864	.2270	1.0356	20
50	0.5381	.5125	.7097	.8587	.9338	.5969	.7759	1.6753	.2241	1.0327	10
31° 00′	0.5411	.5150	9.7118	.8572	9.9331	.6009	9.7788	1.6643	0.2212	1.0297	59° 00′
10	0.5440	.5175	.7139	.8557	.9323	.6048	.7816	1.6534	.2184	1.0268	50
20	0.5469	.5200	.7160	.8542	.9315	.6088	.7845	1.6426	.2155	1.0239	40
30	0.5498	.5225	.7181	.8526	.9308	.6128	.7873	1.6319	.2127	1.0210	30
40	0.5527	.5250	.7201	.8511	.9300	.6168	.7902	1.6212	.2098	1.0181	20
50	0.5556	.5275	.7222	.8496	.9292	.6208	.7930	1.6107	.2070	1.0152	10
32° 00′	0.5585	.5299	9.7242	.8480	9.9284	.6249	9.7958	1.6003	0.2042	1.0123	58° 00′
10	0.5614	.5324	.7262	.8465	.9276	.6289	.7986	1.5900	.2014	1.0094	50
20	0.5643	.5348	.7282	.8450	.9268	.6330	.8014	1.5798	.1986	1.0065	40
30	0.5672	.5373	.7302	.8434	.9260	.6371	.8042	1.5697	.1958	1.0036	30
40	0.5701	.5398	.7322	.8418	.9252	.6412	.8070	1.5597	.1930	1.0007	20
50	0.5730	.5422	.7342	.8403	.9244	.6453	.8097	1.5497	.1903	0.9977	10
33° 00′	0.5760	.5446	9.7361	.8387	9.9236	.6494	9.8125	1.5399	0.1875	0.9948	57° 00′
10	0.5789	.5471	.7380	.8371	.9228	.6536	.8153	1.5301	.1847	0.9919	50
20	0.5818	.5495	.7400	.8355	.9219	.6577	.8180	1.5204	.1820	0.9890	40
30	0.5847	.5519	.7419	.8339	.9211	.6619	.8208	1.5108	.1792	0.9861	30
40	0.5876	.5544	.7438	.8323	.9203	.6661	.8235	1.5013	.1765	0.9832	20
50	0.5905	.5568	.7457	.8307	.9194	.6703	.8263	1.4919	.1737	0.9803	10
34° 00′	0.5934	.5592	9.7476	.8290	9.9186	.6745	9.8290	1.4826	0.1710	0.9774	56° 00′
10	0.5963	.5616	.7494	.8274	.9177	.6787	.8317	1.4733	.1683	0.9745	50
20	0.5992	.5640	.7513	.8258	.9169	.6830	.8344	1.4641	.1656	0.9716	40
30	0.6021	.5664	.7531	.8241	.9160	.6873	.8371	1.4550	.1629	0.9687	30
40	0.6050	.5688	.7550	.8225	.9151	.6916	.8398	1.4460	.1602	0.9657	20
50	0.6080	.5712	.7568	.8208	.9142	.6959	.8425	1.4370	.1575	0.9628	10
35° 00′	0.6109	.5736	9.7586	.8192	9.9134	.7002	9.8452	1.4281	0.1548	0.9599	55° 00′
10	0.6138	.5760	.7604	.8175	.9125	.7046	.8479	1.4193	.1521	0.9570	50
20	0.6167	.5783	.7622	.8158	.9116	.7089	.8506	1.4106	.1494	0.9541	40
30	0.6196	.5807	.7640	.8141	.9107	.7133	.8533	1.4019	.1467	0.9512	30
40	0.6225	.5831	.7657	.8124	.9098	.7177	.8559	1.3934	.1441	0.9483	20
50	0.6254	.5854	.7675	.8107	.9089	.7221	.8586	1.3848	.1414	0.9454	10
36° 00′	0.6283	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	0.9425	54° 00′
		Nat.	Log.*	Nat.	Log.*	Nat.	Log.*	Nat.	Log.		
		Cosines		Sines		Cotangents		Tangents		Radians	Degrees

TRIGONOMETRIC FUNCTIONS (*continued*)

Annex −10 in columns marked*. (For 0°.1 intervals, see pp. 46–51)

Degrees	Radians	Sines Nat.	Sines Log.*	Cosines Nat.	Cosines Log.*	Tangents Nat.	Tangents Log.*	Cotangents Nat.	Cotangents Log.		
36° 00′	0.6283	.5878	9.7692	.8090	9.9080	.7265	9.8613	1.3764	0.1387	0.9425	54° 00′
10	0.6312	.5901	.7710	.8073	.9070	.7310	.8639	1.3680	.1361	0.9396	50
20	0.6341	.5925	.7727	.8056	.9061	.7355	.8666	1.3597	.1334	0.9367	40
30	0.6370	.5948	.7744	.8039	.9052	.7400	.8692	1.3514	.1308	0.9338	30
40	0.6400	.5972	.7761	.8021	.9042	.7445	.8718	1.3432	.1282	0.9308	20
50	0.6429	.5995	.7778	.8004	.9033	.7490	.8745	1.3351	.1255	0.9279	10
37° 00′	0.6458	.6018	9.7795	.7986	9.9023	.7536	9.8771	1.3270	0.1229	0.9250	53° 00′
10	0.6487	.6041	.7811	.7969	.9014	.7581	.8797	1.3190	.1203	0.9221	50
20	0.6516	.6065	.7828	.7951	.9004	.7627	.8824	1.3111	.1176	0.9192	40
30	0.6545	.6088	.7844	.7934	.8995	.7673	.8850	1.3032	.1150	0.9163	30
40	0.6574	.6111	.7861	.7916	.8985	.7720	.8876	1.2954	.1124	0.9134	20
50	0.6603	.6134	.7877	.7898	.8975	.7766	.8902	1.2876	.1098	0.9105	10
38° 00′	0.6632	.6157	9.7893	.7880	9.8965	.7813	9.8928	1.2799	0.1072	0.9076	52° 00′
10	0.6661	.6180	.7910	.7862	.8955	.7860	.8954	1.2723	.1046	0.9047	50
20	0.6690	.6202	.7926	.7844	.8945	.7907	.8980	1.2647	.1020	0.9018	40
30	0.6720	.6225	.7941	.7826	.8935	.7954	.9006	1.2572	.0994	0.8988	30
40	0.6749	.6248	.7957	.7808	.8925	.8002	.9032	1.2497	.0968	0.8959	20
50	0.6778	.6271	.7973	.7790	.8915	.8050	.9058	1.2423	.0942	0.8930	10
39° 00′	0.6807	.6293	9.7989	.7771	9.8905	.8098	9.9084	1.2349	0.0916	0.8901	51° 00′
10	0.6836	.6316	.8004	.7753	.8895	.8146	.9110	1.2276	.0890	0.8872	50
20	0.6865	.6338	.8020	.7735	.8884	.8195	.9135	1.2203	.0865	0.8843	40
30	0.6894	.6361	.8035	.7716	.8874	.8243	.9161	1.2131	.0839	0.8814	30
40	0.6923	.6383	.8050	.7698	.8864	.8292	.9187	1.2059	.0813	0.8785	20
50	0.6952	.6406	.8066	.7679	.8853	.8342	.9212	1.1988	.0788	0.8756	10
40° 00′	0.6981	.6428	9.8081	.7660	9.8843	.8391	9.9238	1.1918	0.0762	0.8727	50° 00′
10	0.7010	.6450	.8096	.7642	.8832	.8441	.9264	1.1847	.0736	0.8698	50
20	0.7039	.6472	.8111	.7623	.8821	.8491	.9289	1.1778	.0711	0.8668	40
30	0.7069	.6494	.8125	.7604	.8810	.8541	.9315	1.1708	.0685	0.8639	30
40	0.7098	.6517	.8140	.7585	.8800	.8591	.9341	1.1640	.0659	0.8610	20
50	0.7127	.6539	.8155	.7566	.8789	.8642	.9366	1.1571	.0634	0.8581	10
41° 00′	0.7156	.6561	9.8169	.7547	9.8778	.8693	9.9392	1.1504	0.0608	0.8552	49° 00′
10	0.7185	.6583	.8184	.7528	.8767	.8744	.9417	1.1436	.0583	0.8523	50
20	0.7214	.6604	.8198	.7509	.8756	.8796	.9443	1.1369	.0557	0.8494	40
30	0.7243	.6626	.8213	.7490	.8745	.8847	.9468	1.1303	.0532	0.8465	30
40	0.7272	.6648	.8227	.7470	.8733	.8899	.9494	1.1237	.0506	0.8436	20
50	0.7301	.6670	.8241	.7451	.8722	.8952	.9519	1.1171	.0481	0.8407	10
42° 00′	0.7330	.6691	9.8255	.7431	9.8711	.9004	9.9544	1.1106	0.0456	0.8378	48° 00′
10	0.7359	.6713	.8269	.7412	.8699	.9057	.9570	1.1041	.0430	0.8348	50
20	0.7389	.6734	.8283	.7392	.8688	.9110	.9595	1.0977	.0405	0.8319	40
30	0.7418	.6756	.8297	.7373	.8676	.9163	.9621	1.0913	.0379	0.8290	30
40	0.7447	.6777	.8311	.7353	.8665	.9217	.9646	1.0850	.0354	0.8261	20
50	0.7476	.6799	.8324	.7333	.8653	.9271	.9671	1.0786	.0329	0.8232	10
43° 00′	0.7505	.6820	9.8338	.7314	9.8641	.9325	9.9697	1.0724	0.0303	0.8203	47° 00′
10	0.7534	.6841	.8351	.7294	.8629	.9380	.9722	1.0661	.0278	0.8174	50
20	0.7563	.6862	.8365	.7274	.8618	.9435	.9747	1.0599	.0253	0.8145	40
30	0.7592	.6884	.8378	.7254	.8606	.9490	.9772	1.0538	.0228	0.8116	30
40	0.7621	.6905	.8391	.7234	.8594	.9545	.9798	1.0477	.0202	0.8087	20
50	0.7650	.6926	.8405	.7214	.8582	.9601	.9823	1.0416	.0177	0.8058	10
44° 00′	0.7679	.6947	9.8418	.7193	9.8569	.9657	9.9848	1.0355	0.0152	0.8029	46° 00′
10	0.7709	.6967	.8431	.7173	.8557	.9713	.9874	1.0295	.0126	0.7999	50
20	0.7738	.6988	.8444	.7153	.8545	.9770	.9899	1.0235	.0101	0.7970	40
30	0.7767	.7009	.8457	.7133	.8532	.9827	.9924	1.0176	.0076	0.7941	30
40	0.7796	.7030	.8469	.7112	.8520	.9884	.9949	1.0117	.0051	0.7912	20
50	0.7825	.7050	.8482	.7092	.8507	.9942	.9975	1.0058	.0025	0.7883	10
45° 00′	0.7854	.7071	9.8495	.7071	9.8495	1.0000	0.0000	1.0000	0.0000	0.7854	45° 00′
		Nat.	Log.*	Nat.	Log.*	Nat.	Log.*	Nat.	Log.		
		Cosines		Sines		Cotangents		Tangents		Radians	Degrees

EXPONENTIALS [e^n and e^{-n}]

n	e^n	Diff.	n	e^n	Diff.	n	e^n	n	e^{-n}	Diff.	n	e^{-n}	n	e^{-n}
0.00	1.000	10	**0.50**	1.649	16	**1.0**	2.718*	**0.00**	1.000	−10	**0.50**	.607	**1.0**	.368*
.01	1.010	10	.51	1.665	17	.1	3.004	.01	0.990	−10	.51	.600	.1	.333
.02	1.020	10	.52	1.682	17	.2	3.320	.02	.980	−10	.52	.595	.2	.301
.03	1.030	11	.53	1.699	17	.3	3.669	.03	.970	−9	.53	.589	.3	.273
.04	1.041	10	.54	1.716	17	.4	4.055	.04	.961	−10	.54	.583	.4	.247
0.05	1.051	11	0.55	1.733	18	1.5	4.482	0.05	.951	−9	0.55	.577	1.5	.223
.06	1.062	11	.56	1.751	17	.6	4.953	.06	.942	−10	.56	.571	.6	.202
.07	1.073	10	.57	1.768	18	.7	5.474	.07	.932	−9	.57	.566	.7	.183
.08	1.083	11	.58	1.786	18	.8	6.050	.08	.923	−9	.58	.560	.8	.165
.09	1.094	11	.59	1.804	18	.9	6.686	.09	.914	−9	.59	.554	.9	.150
0.10	1.105	11	**0.60**	1.822	18	**2.0**	7.389	**0.10**	.905	−9	**0.60**	.549	**2.0**	.135
.11	1.116	11	.61	1.840	19	.1	8.166	.11	.896	−9	.61	.543	.1	.122
.12	1.127	12	.62	1.859	19	.2	9.025	.12	.887	−9	.62	.538	.2	.111
.13	1.139	11	.63	1.878	18	.3	9.974	.13	.878	−9	.63	.533	.3	.100
.14	1.150	12	.64	1.896	20	.4	11.02	.14	.869	−8	.64	.527	.4	.0907
0.15	1.162	12	0.65	1.916	19	2.5	12.18	0.15	.861	−9	0.65	.522	2.5	.0821
.16	1.174	11	.66	1.935	19	.6	13.46	.16	.852	−8	.66	.517	.6	.0743
.17	1.185	12	.67	1.954	20	.7	14.88	.17	.844	−9	.67	.512	.7	.0672
.18	1.197	12	.68	1.974	20	.8	16.44	.18	.835	−8	.68	.507	.8	.0608
.19	1.209	12	.69	1.994	20	.9	18.17	.19	.827	−8	.69	.502	.9	.0550
0.20	1.221	13	**0.70**	2.014	20	**3.0**	20.09	**0.20**	.819	−8	**0.70**	.497	**3.0**	.0498
.21	1.234	12	.71	2.034	20	.1	22.20	.21	.811	−8	.71	.492	.1	.0450
.22	1.246	13	.72	2.054	21	.2	24.53	.22	.803	−8	.72	.487	.2	.0408
.23	1.259	12	.73	2.075	21	.3	27.11	.23	.795	−8	.73	.482	.3	.0369
.24	1.271	13	.74	2.096	21	.4	29.96	.24	.787	−8	.74	.477	.4	.0334
0.25	1.284	13	0.75	2.117	21	3.5	33.12	0.25	.779	−8	0.75	.472	3.5	.0302
.26	1.297	13	.76	2.138	22	.6	36.60	.26	.771	−8	.76	.468	.6	.0273
.27	1.310	13	.77	2.160	21	.7	40.45	.27	.763	−7	.77	.463	.7	.0247
.28	1.323	13	.78	2.181	22	.8	44.70	.28	.756	−8	.78	.458	.8	.0224
.29	1.336	14	.79	2.203	23	.9	49.40	.29	.748	−7	.79	.454	.9	.0202
0.30	1.350	13	**0.80**	2.226	22	**4.0**	54.60	**0.30**	.741	−8	**0.80**	.449	**4.0**	.0183
.31	1.363	14	.81	2.248	22	.1	60.34	.31	.733	−7	.81	.445	.1	.0166
.32	1.377	14	.82	2.270	23	.2	66.69	.32	.726	−7	.82	.440	.2	.0150
.33	1.391	14	.83	2.293	23	.3	73.70	.33	.719	−7	.83	.436	.3	.0136
.34	1.405	14	.84	2.316	24	.4	81.45	.34	.712	−7	.84	.432	.4	.0123
0.35	1.419	14	0.85	2.340	23	4.5	90.02	0.35	.705	−7	0.85	.427	4.5	.0111
.36	1.433	15	.86	2.363	24			.36	.698	−7	.86	.423		
.37	1.448	14	.87	2.387	24	**5.0**	148.4	.37	.691	−7	.87	.419	**5.0**	.00674
.38	1.462	15	.88	2.411	24	6.0	403.4	.38	.684	−7	.88	.415	6.0	.00248
.39	1.477	15	.89	2.435	25	7.0	1097.	.39	.677	−7	.89	.411	7.0	.000912
0.40	1.492	15	**0.90**	2.460	24	8.0	2981.	**0.40**	.670	−6	**0.90**	.407	8.0	.000335
.41	1.507	15	.91	2.484	25	9.0	8103.	.41	.664	−7	.91	.403	9.0	.000123
.42	1.522	15	.92	2.509	26	**10.0**	22026.	.42	.657	−6	.92	.399	**10.0**	.000045
.43	1.537	16	.93	2.535	25	$\pi/2$	4.810	.43	.651	−7	.93	.395	$\pi/2$	.208
.44	1.553	15	.94	2.560	26	$2\pi/2$	23.14	.44	.644	−6	.94	.391	$2\pi/2$	.0432
0.45	1.568	16	0.95	2.586	26	$3\pi/2$	111.3	0.45	.638	−7	0.95	.387	$3\pi/2$	.00898
.46	1.584	16	.96	2.612	26	$4\pi/2$	535.5	.46	.631	−6	.96	.383	$4\pi/2$	.00187
.47	1.600	16	.97	2.638	26	$5\pi/2$	2576.	.47	.625	−6	.97	.379	$5\pi/2$	.000388
.48	1.616	16	.98	2.664	27	$6\pi/2$	12392.	.48	.619	−6	.98	.375	$6\pi/2$	.000081
.49	1.632	17	.99	2.691	27	$7\pi/2$	59610.	.49	.613	−6	.99	.372	$7\pi/2$	.000017
0.50	1.649		**1.00**	2.718		$8\pi/2$	286751.	**0.50**	0.607		**1.00**	.368	$8\pi/2$	.000003

*NOTE: Do not interpolate in this column.

$e = 2.71828$ $1/e = 0.367879$ $\log_{10} e = 0.4343$ $1/(0.4343) = 2.3026$

$\log_{10}(0.4343) = \bar{1}.6378$ $\log_{10}(e^n) = n(0.4343)$

For table of multiples of 0.4343, see p. 62. Graphs, p. 174.

HYPERBOLIC LOGARITHMS

These two pages give the natural (hyperbolic, or Napierian) logarithms ($\log_e$) of numbers between 1 and 10, correct to four places. Moving the decimal point n places to the right [or left] in the number is equivalent to adding n times 2.3026 [or n times $\bar{3}.6974$] to the logarithm. Base $e = 2.71828+$

n	n (2.3026)	n (0.6974–3)
1	2.3026	0.6974–3
2	4.6052	0.3948–5
3	6.9078	0.0922–7
4	9.2103	0.7897–10
5	11.5129	0.4871–12
6	13.8155	0.1845–14
7	16.1181	0.8819–17
8	18.4207	0.5793–19
9	20.7233	0.2767–21

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	0.0000	0100	0198	0296	0392	0488	0583	0677	0770	0862	95
1.1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	87
1.2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	80
1.3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	74
1.4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	69
1.5	0.4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	65
1.6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	61
1.7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	57
1.8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	54
1.9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	51
2.0	0.6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	49
2.1	7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	47
2.2	7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	44
2.3	8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	43
2.4	8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	41
2.5	0.9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	39
2.6	9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	38
2.7	0.9933	9969	*0006	*0043	*0080	*0116	*0152	*0188	*0225	*0260	36
2.8	1.0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	35
2.9	0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	34
3.0	1.0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	33
3.1	1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	32
3.2	1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	31
3.3	1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	30
3.4	2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	29
3.5	1.2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	28
3.6	2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	27
3.7	3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	27
3.8	3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	26
3.9	3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	25
4.0	1.3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	25
4.1	4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	24
4.2	4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	23
4.3	4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	23
4.4	4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	22
4.5	1.5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	22
4.6	5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	21
4.7	5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	21
4.8	5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	20
4.9	5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	20

$\log_e x = (2.3026) \log_{10} x$ $\log_{10} x = (0.4343) \log_e x$

where $2.3026 = \log_e 10$ and $0.4343 = \log_{10} e$ (see p. 62). For graphs, see p. 174.

HYPERBOLIC LOGARITHMS (*continued*)

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
5.0	1.6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	20
5.1	6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	19
5.2	6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	19
5.3	6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	18
5.4	6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	18
5.5	1.7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	18
5.6	7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	18
5.7	7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	17
5.8	7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	17
5.9	7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	17
6.0	1.7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	16
6.1	8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	16
6.2	8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	16
6.3	8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	16
6.4	8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	15
6.5	1.8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	15
6.6	8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	15
6.7	9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	15
6.8	9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	15
6.9	9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	14
7.0	1.9459	9473	9488	9502	9516	9530	9544	9559	9573	9587	14
7.1	9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	14
7.2	9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	14
7.3	1.9879	9892	9906	9920	9933	9947	9961	9974	9988	*0001	13
7.4	2.0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	13
7.5	2.0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	13
7.6	0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	13
7.7	0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	13
7.8	0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	13
7.9	0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	12
8.0	2.0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	12
8.1	0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	12
8.2	1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	12
8.3	1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	12
8.4	1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	12
8.5	2.1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	12
8.6	1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	12
8.7	1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	11
8.8	1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	11
8.9	1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	11
9.0	2.1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	11
9.1	2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	11
9.2	2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	11
9.3	2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	11
9.4	2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	11
9.5	2.2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	10
9.6	2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	10
9.7	2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	10
9.8	2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	10
9.9	2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	10
10.0	2.3026										

Moving the decimal point n places to the right [or left] in the number requires adding n times 2.3026 [or n times (0.6974−3)] in the body of the table. See auxiliary table of multiples on top of the preceding page.

HYPERBOLIC SINES [$\sinh x = \frac{1}{2}(e^x - e^{-x})$]

x	0	1	2	3	4	5	6	7	8	9	Avg. diff.
0.0	.0000	.0100	.0200	.0300	.0400	.0500	.0600	.0701	.0801	.0901	100
1	.1002	.1102	.1203	.1304	.1405	.1506	.1607	.1708	.1810	.1911	101
2	.2013	.2115	.2218	.2320	.2423	.2526	.2629	.2733	.2837	.2941	103
3	.3045	.3150	.3255	.3360	.3466	.3572	.3678	.3785	.3892	.4000	106
4	.4108	.4216	.4325	.4434	.4543	.4653	.4764	.4875	.4986	.5098	110
0.5	.5211	.5324	.5438	.5552	.5666	.5782	.5897	.6014	.6131	.6248	116
6	.6367	.6485	.6605	.6725	.6846	.6967	.7090	.7213	.7336	.7461	122
7	.7586	.7712	.7838	.7966	.8094	.8223	.8353	.8484	.8615	.8748	130
8	.8881	.9015	.9150	.9286	.9423	.9561	.9700	.9840	.9981	1.012	138
9	1.027	1.041	1.055	1.070	1.085	1.099	1.114	1.129	1.145	1.160	15
1.0	1.175	1.191	1.206	1.222	1.238	1.254	1.270	1.286	1.303	1.319	16
1	1.336	1.352	1.369	1.386	1.403	1.421	1.438	1.456	1.474	1.491	17
2	1.509	1.528	1.546	1.564	1.583	1.602	1.621	1.640	1.659	1.679	19
3	1.698	1.718	1.738	1.758	1.779	1.799	1.820	1.841	1.862	1.883	21
4	1.904	1.926	1.948	1.970	1.992	2.014	2.037	2.060	2.083	2.106	22
1.5	2.129	2.153	2.177	2.201	2.225	2.250	2.274	2.299	2.324	2.350	25
6	2.376	2.401	2.428	2.454	2.481	2.507	2.535	2.562	2.590	2.617	27
7	2.646	2.674	2.703	2.732	2.761	2.790	2.820	2.850	2.881	2.911	30
8	2.942	2.973	3.005	3.037	3.069	3.101	3.134	3.167	3.200	3.234	33
9	3.268	3.303	3.337	3.372	3.408	3.443	3.479	3.516	3.552	3.589	36
2.0	3.627	3.665	3.703	3.741	3.780	3.820	3.859	3.899	3.940	3.981	39
1	4.022	4.064	4.106	4.148	4.191	4.234	4.278	4.322	4.367	4.412	44
2	4.457	4.503	4.549	4.596	4.643	4.691	4.739	4.788	4.837	4.887	48
3	4.937	4.988	5.039	5.090	5.142	5.195	5.248	5.302	5.356	5.411	53
4	5.466	5.522	5.578	5.635	5.693	5.751	5.810	5.869	5.929	5.989	58
2.5	6.050	6.112	6.174	6.237	6.300	6.365	6.429	6.495	6.561	6.627	64
6	6.695	6.763	6.831	6.901	6.971	7.042	7.113	7.185	7.258	7.332	71
7	7.406	7.481	7.557	7.634	7.711	7.789	7.868	7.948	8.028	8.110	79
8	8.192	8.275	8.359	8.443	8.529	8.615	8.702	8.790	8.879	8.969	87
9	9.060	9.151	9.244	9.337	9.431	9.527	9.623	9.720	9.819	9.918	96
3.0	10.02	10.12	10.22	10.32	10.43	10.53	10.64	10.75	10.86	10.97	11
1	11.08	11.19	11.30	11.42	11.53	11.65	11.76	11.88	12.00	12.12	12
2	12.25	12.37	12.49	12.62	12.75	12.88	13.01	13.14	13.27	13.40	13
3	13.54	13.67	13.81	13.95	14.09	14.23	14.38	14.52	14.67	14.82	14
4	14.97	15.12	15.27	15.42	15.58	15.73	15.89	16.05	16.21	16.38	16
3.5	16.54	16.71	16.88	17.05	17.22	17.39	17.57	17.74	17.92	18.10	17
6	18.29	18.47	18.66	18.84	19.03	19.22	19.42	19.61	19.81	20.01	19
7	20.21	20.41	20.62	20.83	21.04	21.25	21.46	21.68	21.90	22.12	21
8	22.34	22.56	22.79	23.02	23.25	23.49	23.72	23.96	24.20	24.45	24
9	24.69	24.94	25.19	25.44	25.70	25.96	26.22	26.48	26.75	27.02	26
4.0	27.29	27.56	27.84	28.12	28.40	28.69	28.98	29.27	29.56	29.86	29
1	30.16	30.47	30.77	31.08	31.39	31.71	32.03	32.35	32.68	33.00	32
2	33.34	33.67	34.01	34.35	34.70	35.05	35.40	35.75	36.11	36.48	35
3	36.84	37.21	37.59	37.97	38.35	38.73	39.12	39.52	39.91	40.31	39
4	40.72	41.13	41.54	41.96	42.38	42.81	43.24	43.67	44.11	44.56	43
4.5	45.00	45.46	45.91	46.37	46.84	47.31	47.79	48.27	48.75	49.24	47
6	49.74	50 24	50.74	51.25	51.77	52.29	52.81	53.34	53.88	54.42	52
7	54.97	55.52	56.08	56.64	57.21	57.79	58.37	58.96	59.55	60.15	58
8	60.75	61.36	61.98	62.60	63.23	63.87	64.51	65.16	65.81	66.47	64
9	67.14	67.82	68.50	69.19	69.88	70.58	71.29	72.01	72.73	73.46	71
5.0	74.20										

If $x > 5$, $\sinh x = \frac{1}{2}(e^x)$ and $\log_{10} \sinh x = (0.4343)x + 0.6990 - 1$, correct to four significant figures. For table of multiples of 0.4343, see p. 62. Graphs, p. 174.

HYPERBOLIC COSINES [$\cosh x = \frac{1}{2}(e^x + e^{-x})$]

x	0	1	2	3	4	5	6	7	8	9	Avg. diff.
0.0	1.000	1.000	1.000	1.000	1.001	1.001	1.002	1.002	1.003	1.004	1
1	1.005	1.006	1.007	1.008	1.010	1.011	1.013	1.014	1.016	1.018	2
2	1.020	1.022	1.024	1.027	1.029	1.031	1.034	1.037	1.039	1.042	3
3	1.045	1.048	1.052	1.055	1.058	1.062	1.066	1.069	1.073	1.077	4
4	1.081	1.085	1.090	1.094	1.098	1.103	1.108	1.112	1.117	1.122	5
0.5	1.128	1.133	1.138	1.144	1.149	1.155	1.161	1.167	1.173	1.179	6
6	1.185	1.192	1.198	1.205	1.212	1.219	1.226	1.233	1.240	1.248	7
7	1.255	1.263	1.271	1.278	1.287	1.295	1.303	1.311	1.320	1.329	8
8	1.337	1.346	1.355	1.365	1.374	1.384	1.393	1.403	1.413	1.423	10
9	1.433	1.443	1.454	1.465	1.475	1.486	1.497	1.509	1.520	1.531	11
1.0	1.543	1.555	1.567	1.579	1.591	1.604	1.616	1.629	1.642	1.655	13
1	1.669	1.682	1.696	1.709	1.723	1.737	1.752	1.766	1.781	1.796	14
2	1.811	1.826	1.841	1.857	1.872	1.888	1.905	1.921	1.937	1.954	16
3	1.971	1.988	2.005	2.023	2.040	2.058	2.076	2.095	2.113	2.132	18
4	2.151	2.170	2.189	2.209	2.229	2.249	2.269	2.290	2.310	2.331	20
1.5	2.352	2.374	2.395	2.417	2.439	2.462	2.484	2.507	2.530	2.554	23
6	2.577	2.601	2.625	2.650	2.675	2.700	2.725	2.750	2.776	2.802	25
7	2.828	2.855	2.882	2.909	2.936	2.964	2.992	3.021	3.049	3.078	28
8	3.107	3.137	3.167	3.197	3.228	3.259	3.290	3.321	3.353	3.385	31
9	3.418	3.451	3.484	3.517	3.551	3.585	3.620	3.655	3.690	3.726	34
2.0	3.762	3.799	3.835	3.873	3.910	3.948	3.987	4.026	4.065	4.104	38
1	4.144	4.185	4.226	4.267	4.309	4.351	4.393	4.436	4.480	4.524	42
2	4.568	4.613	4.658	4.704	4.750	4.797	4.844	4.891	4.939	4.988	47
3	5.037	5.087	5.137	5.188	5.239	5.290	5.343	5.395	5.449	5.503	52
4	5.557	5.612	5.667	5.723	5.780	5.837	5.895	5.954	6.013	6.072	58
2.5	6.132	6.193	6.255	6.317	6.379	6.443	6.507	6.571	6.636	6.702	64
6	6.769	6.836	6.904	6.973	7.042	7.112	7.183	7.255	7.327	7.400	70
7	7.473	7.548	7.623	7.699	7.776	7.853	7.932	8.011	8.091	8.171	78
8	8.253	8.335	8.418	8.502	8.587	8.673	8.759	8.847	8.935	9.024	86
9	9.115	9.206	9.298	9.391	9.484	9.579	9.675	9.772	9.869	9.968	95
3.0	10.07	10.17	10.27	10.37	10.48	10.58	10.69	10.79	10.90	11.01	11
1	11.12	11.23	11.35	11.46	11.57	11.69	11.81	11.92	12.04	12.16	12
2	12.29	12.41	12.53	12.66	12.79	12.91	13.04	13.17	13.31	13.44	13
3	13.57	13.71	13.85	13.99	14.13	14.27	14.41	14.56	14.70	14.85	14
4	15.00	15.15	15.30	15.45	15.61	15.77	15.92	16.08	16.25	16.41	16
3.5	16.57	16.74	16.91	17.08	17.25	17.42	17.60	17.77	17.95	18.13	17
6	18.31	18.50	18.68	18.87	19.06	19.25	19.44	19.64	19.84	20.03	19
7	20.24	20.44	20.64	20.85	21.06	21.27	21.49	21.70	21.92	22.14	21
8	22.36	22.59	22.81	23.04	23.27	23.51	23.74	23.98	24.22	24.47	23
9	24.71	24.96	25.21	25.46	25.72	25.98	26.24	26.50	26.77	27.04	26
4.0	27.31	27.58	27.86	28.14	28.42	28.71	29.00	29.29	29.58	29.88	29
1	30.18	30.48	30.79	31.10	31.41	31.72	32.04	32.37	32.69	33.02	32
2	33.35	33.69	34.02	34.37	34.71	35.06	35.41	35.77	36.13	36.49	35
3	36.86	37.23	37.60	37.98	38.36	38.75	39.13	39.53	39.93	40.33	39
4	40.73	41.14	41.55	41.97	42.39	42.82	43.25	43.68	44.12	44.57	43
4.5	45.01	45.47	45.92	46.38	46.85	47.32	47.80	48.28	48.76	49.25	47
6	49.75	50.25	50.75	51.26	51.78	52.30	52.82	53.35	53.89	54.43	52
7	54.98	55.53	56.09	56.65	57.22	57.80	58.38	58.96	59.56	60.15	58
8	60.76	61.37	61.99	62.61	63.24	63.87	64.52	65.16	65.82	66.48	64
9	67.15	67.82	68.50	69.19	69.89	70.59	71.30	72.02	72.74	73.47	71
5.0	74.21										

If $x > 5$, $\cosh x = \frac{1}{2}(e^x)$ and $\log_{10} \cosh x = (0.4343)x + 0.6990 - 1$, correct to four significant figures. For table of multiples of 0.4343, see p. 62. Graphs, p. 174.

HYPERBOLIC TANGENTS [$\tanh x = (e^x - e^{-x})/(e^x + e^{-x}) = \sinh x/\cosh x$]

x	0	1	2	3	4	5	6	7	8	9	Avg. diff.
0.0	.0000	.0100	.0200	.0300	.0400	.0500	.0599	.0699	.0798	.0898	100
1	.0997	.1096	.1194	.1293	.1391	.1489	.1587	.1684	.1781	.1878	98
2	.1974	.2070	.2165	.2260	.2355	.2449	.2543	.2636	.2729	.2821	94
3	.2913	.3004	.3095	.3185	.3275	.3364	.3452	.3540	.3627	.3714	89
4	.3800	.3885	.3969	.4053	.4136	.4219	.4301	.4382	.4462	.4542	82
0.5	.4621	.4700	.4777	.4854	.4930	.5005	.5080	.5154	.5227	.5299	75
6	.5370	.5441	.5511	.5581	.5649	.5717	.5784	.5850	.5915	.5980	67
7	.6044	.6107	.6169	.6231	.6291	.6352	.6411	.6469	.6527	.6584	60
8	.6640	.6696	.6751	.6805	.6858	.6911	.6963	.7014	.7064	.7114	52
9	.7163	.7211	.7259	.7306	.7352	.7398	.7443	.7487	.7531	.7574	45
1.0	.7616	.7658	.7699	.7739	.7779	.7818	.7857	.7895	.7932	.7969	39
1	.8005	.8041	.8076	.8110	.8144	.8178	.8210	.8243	.8275	.8306	33
2	.8337	.8367	.8397	.8426	.8455	.8483	.8511	.8538	.8565	.8591	28
3	.8617	.8643	.8668	.8693	.8717	.8741	.8764	.8787	.8810	.8832	24
4	.8854	.8875	.8896	.8917	.8937	.8957	.8977	.8996	.9015	.9033	20
1.5	.9052	.9069	.9087	.9104	.9121	.9138	.9154	.9170	.9186	.9202	17
6	.9217	.9232	.9246	.9261	.9275	.9289	.9302	.9316	.9329	.9342	14
7	.9354	.9367	.9379	.9391	.9402	.9414	.9425	.9436	.9447	.9458	11
8	.9468	.9478	.9488	.9498	.9508	.9518	.9527	.9536	.9545	.9554	9
9	.9562	.9571	.9579	.9587	.9595	.9603	.9611	.9619	.9626	.9633	8
2.0	.9640	.9647	.9654	.9661	.9668	.9674	.9680	.9687	.9693	.9699	6
1	.9705	.9710	.9716	.9722	.9727	.9732	.9738	.9743	.9748	.9753	5
2	.9757	.9762	.9767	.9771	.9776	.9780	.9785	.9789	.9793	.9797	4
3	.9801	.9805	.9809	.9812	.9816	.9820	.9823	.9827	.9830	.9834	4
4	.9837	.9840	.9843	.9846	.9849	.9852	.9855	.9858	.9861	.9863	3
2.5	.9866	.9869	.9871	.9874	.9876	.9879	.9881	.9884	.9886	.9888	2
6	.9890	.9892	.9895	.9897	.9899	.9901	.9903	.9905	.9906	.9908	2
7	.9910	.9912	.9914	.9915	.9917	.9919	.9920	.9922	.9923	.9925	2
8	.9926	.9928	.9929	.9931	.9932	.9933	.9935	.9936	.9937	.9938	1
2.9	.9940	.9941	.9942	.9943	.9944	.9945	.9946	.9947	.9949	.9950	1
3.	.9951	.9959	.9967	.9973	.9978	.9982	.9985	.9988	.9990	.9992	4
4.	.9993	.9995	.9996	.9996	.9997	.9998	.9998	.9998	.9999	.9999	1
5.	.9999	If $x > 5$, $\tanh x = 1.0000$ to four decimal places. Graphs, p. 174.									

MULTIPLES OF 0.4343 ($0.43429448 = \log_{10} e$)

x	0	1	2	3	4	5	6	7	8	9
0.	0.0000	0.0434	0.0869	0.1303	0.1737	0.2171	0.2606	0.3040	0.3474	0.3909
1.	0.4343	0.4777	0.5212	0.5646	0.6080	0.6514	0.6949	0.7383	0.7817	0.8252
2.	0.8686	0.9120	0.9554	0.9989	1.0423	1.0857	1.1292	1.1726	1.2160	1.2595
3.	1.3029	1.3463	1.3897	1.4332	1.4766	1.5200	1.5635	1.6069	1.6503	1.6937
4.	1.7372	1.7806	1.8240	1.8675	1.9109	1.9543	1.9978	2.0412	2.0846	2.1280
5.	2.1715	2.2149	2.2583	2.3018	2.3452	2.3886	2.4320	2.4755	2.5189	2.5623
6.	2.6058	2.6492	2.6926	2.7361	2.7795	2.8229	2.8663	2.9098	2.9532	2.9966
7.	3.0401	3.0835	3.1269	3.1703	3.2138	3.2572	3.3006	3.3441	3.3875	3.4309
8.	3.4744	3.5178	3.5612	3.6046	3.6481	3.6915	3.7349	3.7784	3.8218	3.8652
9.	3.9087	3.9521	3.9955	4.0389	4.0824	4.1258	4.1692	4.2127	4.2561	4.2995

MULTIPLES OF 2.3026 ($2.3025851 = 1/0.4343$)

x	0	1	2	3	4	5	6	7	8	9
0.	0.0000	0.2303	0.4605	0.6908	0.9210	1.1513	1.3816	1.6118	1.8421	2.0723
1.	2.3026	2.5328	2.7631	2.9934	3.2236	3.4539	3.6841	3.9144	4.1447	4.3749
2.	4.6052	4.8354	5.0657	5.2959	5.5262	5.7565	5.9867	6.2170	6.4472	6.6775
3.	6.9078	7.1380	7.3683	7.5985	7.8288	8.0590	8.2893	8.5196	8.7498	8.9801
4.	9.2103	9.4406	9.6709	9.9011	10.131	10.362	10.592	10.822	11.052	11.283
5.	11.513	11.743	11.973	12.204	12.434	12.664	12.894	13.125	13.355	13.585
6.	13.816	14.046	14.276	14.506	14.737	14.967	15.197	15.427	15.658	15.888
7.	16.118	16.348	16.579	16.809	17.039	17.269	17.500	17.730	17.960	18.190
8.	18.421	18.651	18.881	19.111	19.342	19.572	19.802	20.032	20.263	20.493
9.	20.723	20.954	21.184	21.414	21.644	21.875	22.105	22.335	22.565	22.796

STANDARD DISTRIBUTION OF RESIDUALS (p. 121)

a = any positive quantity;
y = the number of residuals which are numerically $< a$;
r = the probable error of a single observation;
n = number of observations.

$\frac{a}{r}$	$\frac{y}{n}$	Diff.
0.0	.000	54
1	.054	53
2	.107	53
3	.160	53
4	.213	51
0.5	.264	50
6	.314	49
7	.363	48
8	.411	45
9	.456	44
1.0	.500	42
1	.542	40
2	.582	37
3	.619	36
4	.655	33
1.5	.688	31
6	.719	29
7	.748	27
8	.775	25
9	.800	23
2.0	.823	20
1	.843	19
2	.862	17
3	.879	16
4	.895	13
2.5	.908	13
6	.921	10
7	.931	10
8	.941	9
9	.950	7
3.0	.957	6
1	.963	6
2	.969	5
3	.974	4
4	.978	4
3.5	.982	3
6	.985	2
7	.987	3
8	.990	1
9	.991	2
4.0	.993	6
5.0	.999	

FACTORS FOR COMPUTING PROBABLE ERROR (p. 121)

	Bessel		Peters	
n	$\frac{0.6745}{\sqrt{(n-1)}}$	$\frac{0.6745}{\sqrt{n(n-1)}}$	$\frac{0.8453}{\sqrt{n(n-1)}}$	$\frac{0.8453}{n\sqrt{n-1}}$
2	.6745	.4769	.5978	.4227
3	.4769	.2754	.3451	.1993
4	.3894	.1947	.2440	.1220
5	.3372	.1508	.1890	.0845
6	.3016	.1231	.1543	.0630
7	.2754	.1041	.1304	.0493
8	.2549	.0901	.1130	.0399
9	.2385	.0795	.0996	.0332
10	.2248	.0711	.0891	.0282
11	.2133	.0643	.0806	.0243
12	.2034	.0587	.0736	.0212
13	.1947	.0540	.0677	.0188
14	.1871	.0500	.0627	.0167
15	.1803	.0465	.0583	.0151
16	.1742	.0435	.0546	.0136
17	.1686	.0409	.0513	.0124
18	.1636	.0386	.0483	.0114
19	.1590	.0365	.0457	.0105
20	.1547	.0346	.0434	.0097
21	.1508	.0329	.0412	.0090
22	.1472	.0314	.0393	.0084
23	.1438	.0300	.0376	.0078
24	.1406	.0287	.0360	.0073
25	.1377	.0275	.0345	.0069
26	.1349	.0265	.0332	.0065
27	.1323	.0255	.0319	.0061
28	.1298	.0245	.0307	.0058
29	.1275	.0237	.0297	.0055
30	.1252	.0229	.0287	.0052
31	.1231	.0221	.0277	.0050
32	.1211	.0214	.0268	.0047
33	.1192	.0208	.0260	.0045
34	.1174	.0201	.0252	.0043
35	.1157	.0196	.0245	.0041
36	.1140	.0190	.0238	.0040
37	.1124	.0185	.0232	.0038
38	.1109	.0180	.0225	.0037
39	.1094	.0175	.0220	.0035
40	.1080	.0171	.0214	.0034
45	.1017	.0152	.0190	.0028
50	.0964	.0136	.0171	.0024
55	.0918	.0124	.0155	.0021
60	.0878	.0113	.0142	.0018
65	.0843	.0105	.0131	.0016
70	.0812	.0097	.0122	.0015
75	.0784	.0091	.0113	.0013
80	.0759	.0085	.0106	.0012
85	.0736	.0080	.0100	.0011
90	.0715	.0075	.0094	.0010
95	.0696	.0071	.0089	.0009
100	.0678	.0068	.0085	.0008

COMPOUND INTEREST. AMOUNT OF A GIVEN PRINCIPAL

The amount A at the end of n years of a given principal P placed at compound interest to-day is $A = P \times x$ or $A = P \times y$ or $A = P \times z$, according as the interest (at the rate of r per cent. per annum) is compounded annually, semi-annually, or quarterly; the factor x or y or z being taken from the following tables.

Values of x. (Interest compounded annually; $A = P \times x$.)

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	1.0200	1.0250	1.0300	1.0350	1.0400	1.0450	1.0500	1.0600	1.0700
2	1.0404	1.0506	1.0609	1.0712	1.0816	1.0920	1.1025	1.1236	1.1449
3	1.0612	1.0769	1.0927	1.1087	1.1249	1.1412	1.1576	1.1910	1.2250
4	1.0824	1.1038	1.1255	1.1475	1.1699	1.1925	1.2155	1.2625	1.3108
5	1.1041	1.1314	1.1593	1.1877	1.2167	1.2462	1.2763	1.3382	1.4026
6	1.1262	1.1597	1.1941	1.2293	1.2653	1.3023	1.3401	1.4185	1.5007
7	1.1487	1.1887	1.2299	1.2723	1.3159	1.3609	1.4071	1.5036	1.6058
8	1.1717	1.2184	1.2668	1.3168	1.3686	1.4221	1.4775	1.5938	1.7182
9	1.1951	1.2489	1.3048	1.3629	1.4233	1.4861	1.5513	1.6895	1.8385
10	1.2190	1.2801	1.3439	1.4106	1.4802	1.5530	1.6289	1.7908	1.9672
11	1.2434	1.3121	1.3842	1.4600	1.5395	1.6239	1.7103	1.8983	2.1049
12	1.2682	1.3449	1.4258	1.5111	1.6010	1.6959	1.7959	2.0122	2.2522
13	1.2936	1.3785	1.4685	1.5640	1.6651	1.7722	1.8856	2.1329	2.4098
14	1.3195	1.4130	1.5126	1.6187	1.7317	1.3519	1.9799	2.2609	2.5785
15	1.3459	1.4483	1.5580	1.6753	1.8009	1.9353	2.0789	2.3966	2.7590
16	1.3728	1.4845	1.6047	1.7340	1.8730	2.0224	2.1829	2.5404	2.9522
17	1.4002	1.5216	1.6528	1.7947	1.9479	2.1134	2.2920	2.6928	3.1588
18	1.4282	1.5597	1.7024	1.8575	2.0258	2.2085	2.4066	2.8543	3.3799
19	1.4568	1.5987	1.7535	1.9225	2.1068	2.3079	2.5270	3.0256	3.6165
20	1.4859	1.6386	1.8061	1.9898	2.1911	2.4117	2.6533	3.2071	3.8697
25	1.6406	1.8539	2.0938	2.3632	2.6658	3.0054	3.3864	4.2919	5.4274
30	1.8114	2.0976	2.4273	2.8068	3.2434	3.7453	4.3219	5.7435	7.6123
40	2.2080	2.6851	3.2620	3.9593	4.8010	5.8164	7.0400	10.286	14.974
50	2.6916	3.4371	4.3839	5.5849	7.1067	9.0326	11.467	18.420	29.457
60	3.2810	4.3998	5.8916	7.8781	10.520	14.027	18.679	32.988	57.946

This table is computed from the formula $x = [1 + (r/100)]^n$.

Values of y. (Interest compounded semi-annually; $A = P \times y$.)

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	1.0201	1.0252	1.0302	1.0353	1.0404	1.0455	1.0506	1.0609	1.0712
2	1.0406	1.0509	1.0614	1.0719	1.0824	1.0931	1.1038	1.1255	1.1475
3	1.0615	1.0774	1.0934	1.1097	1.1262	1.1428	1.1597	1.1941	1.2293
4	1.0829	1.1045	1.1265	1.1489	1.1717	1.1948	1.2184	1.2668	1.3168
5	1.1046	1.1323	1.1605	1.1894	1.2190	1.2492	1.2801	1.3439	1.4106
6	1.1268	1.1608	1.1956	1.2314	1.2682	1.3060	1.3449	1.4258	1.5111
7	1.1495	1.1900	1.2318	1.2749	1.3195	1.3655	1.4130	1.5126	1.6187
8	1.1726	1.2199	1.2690	1.3199	1.3728	1.4276	1.4845	1.6047	1.7340
9	1.1961	1.2506	1.3073	1.3665	1.4282	1.4926	1.5597	1.7024	1.8575
10	1.2202	1.2820	1.3469	1.4148	1.4859	1.5605	1.6386	1.8061	1.9898
11	1.2447	1.3143	1.3876	1.4647	1.5460	1.6315	1.7216	1.9161	2.1315
12	1.2697	1.3474	1.4295	1.5164	1.6084	1.7058	1.8087	2.0328	2.2833
13	1.2953	1.3812	1.4727	1.5700	1.6734	1.7834	1.9003	2.1566	2.4460
14	1.3213	1.4160	1.5172	1.6254	1.7410	1.8645	1.9965	2.2879	2.6202
15	1.3478	1.4516	1.5631	1.6828	1.8114	1.9494	2.0976	2.4273	2.8068
16	1.3749	1.4881	1.6103	1.7422	1.8845	2.0381	2.2038	2.5751	3.0067
17	1.4026	1.5256	1.6590	1.8037	1.9607	2.1308	2.3153	2.7319	3.2209
18	1.4308	1.5639	1.7091	1.8674	2.0399	2.2278	2.4325	2.8983	3.4503
19	1.4595	1.6033	1.7608	1.9333	2.1223	2.3292	2.5557	3.0748	3.6960
20	1.4889	1.6436	1.8140	2.0016	2.2080	2.4352	2.6851	3.2620	3.9593
25	1.6446	1.8610	2.1052	2.3808	2.6916	3.0420	3.4371	4.3839	5.5849
30	1.8167	2.1072	2.4432	2.8318	3.2810	3.8001	4.3998	5.8916	7.8781
40	2.2167	2.7015	3.2907	4.0064	4.8754	5.9301	7.2096	10.641	15.676
50	2.7048	3.4634	4.4320	5.6682	7.2446	9.2540	11.814	19.219	31.191
60	3.3004	4.4402	5.9693	8.0192	10.765	14.441	19.358	34.711	62.064

Formula: $y = [1 + (r/200)]^{2n}$.

Values of z. (Interest compounded quarterly; $A = P \times z$; see opposite page)

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	1.0202	1.0252	1.0303	1.0355	1.0406	1.0458	1.0509	1.0614	1.0719
2	1.0407	1.0511	1.0616	1.0722	1.0829	1.0936	1.1045	1.1265	1.1489
3	1.0617	1.0776	1.0938	1.1102	1.1268	1.1437	1.1608	1.1956	1.2314
4	1.0831	1.1048	1.1270	1.1496	1.1726	1.1960	1.2199	1.2690	1.3199
5	1.1049	1.1327	1.1612	1.1903	1.2202	1.2508	1.2820	1.3469	1.4148
6	1.1272	1.1613	1.1964	1.2326	1.2697	1.3080	1.3474	1.4295	1.5164
7	1.1499	1.1906	1.2327	1.2763	1.3213	1.3679	1.4160	1.5172	1.6254
8	1.1730	1.2206	1.2701	1.3215	1.3749	1.4305	1.4881	1.6103	1.7422
9	1.1967	1.2514	1.3086	1.3684	1.4308	1.4959	1.5639	1.7091	1.8674
10	1.2208	1.2830	1.3483	1.4169	1.4889	1.5644	1.6436	1.8140	2.0016
11	1.2454	1.3154	1.3893	1.4672	1.5493	1.6360	1.7274	1.9253	2.1454
12	1.2705	1.3486	1.4314	1.5192	1.6122	1.7108	1.8154	2.0435	2.2996
13	1.2961	1.3826	1.4748	1.5731	1.6777	1.7891	1.9078	2.1689	2.4648
14	1.3222	1.4175	1.5196	1.6288	1.7458	1.8710	2.0050	2.3020	2.6420
15	1.3489	1.4533	1.5657	1.6866	1.8167	1.9566	2.1072	2.4432	2.8318
16	1.3760	1.4900	1.6132	1.7464	1.8905	2.0462	2.2145	2.5931	3.0353
17	1.4038	1.5276	1.6621	1.8083	1.9672	2.1398	2.3274	2.7523	3.2534
18	1.4320	1.5661	1.7126	1.8725	2.0471	2.2378	2.4459	2.9212	3.4872
19	1.4609	1.6056	1.7645	1.9389	2.1302	2.3402	2.5705	3.1004	3.7378
20	1.4903	1.6462	1.8180	2.0076	2.2167	2.4473	2.7015	3.2907	4.0064
25	1.6467	1.8646	2.1111	2.3898	2.7048	3.0609	3.4634	4.4320	5.6682
30	1.8194	2.1121	2.4514	2.8446	3.3004	3.8285	4.4402	5.9693	8.0192
40	2.2211	2.7098	3.3053	4.0306	4.9138	5.9892	7.2980	10.828	16.051
50	2.7115	3.4768	4.4567	5.7110	7.3160	9.3693	11.995	19.643	32.128
60	3.3102	4.4608	6.0092	8.0919	10.893	14.657	19.715	35.633	64.307

Formula: $z = [1 + (r/400)]^{4n}$.

AMOUNT OF AN ANNUITY

The amount S accumulated at the end of n years by a given annual payment Y set aside at the end of each year is $S = Y \times v$, where the factor v is to be taken from the following table. (Interest at r per cent. per annum, compounded annually.)

Values of v

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0200	2.0250	2.0300	2.0350	2.0400	2.0450	2.0500	2.0600	2.0700
3	3.0604	3.0756	3.0909	3.1062	3.1216	3.1370	3.1525	3.1836	3.2149
4	4.1216	4.1525	4.1836	4.2149	4.2465	4.2782	4.3101	4.3746	4.4399
5	5.2040	5.2563	5.3091	5.3625	5.4163	5.4707	5.5256	5.6371	5.7507
6	6.3081	6.3877	6.4684	6.5502	6.6330	6.7169	6.8019	6.9753	7.1533
7	7.4343	7.5474	7.6625	7.7794	7.8983	8.0192	8.1420	8.3938	8.6540
8	8.5830	8.7361	8.8923	9.0517	9.2142	9.3800	9.5491	9.8975	10.260
9	9.7546	9.9545	10.159	10.368	10.583	10.802	11.027	11.491	11.978
10	10.950	11.203	11.464	11.731	12.006	12.288	12.578	13.181	13.816
11	12.169	12.483	12.808	13.142	13.486	13.841	14.207	14.972	15.784
12	13.412	13.796	14.192	14.602	15.026	15.464	15.917	16.870	17.888
13	14.680	15.140	15.618	16.113	16.627	17.160	17.713	18.882	20.141
14	15.974	16.519	17.086	17.677	18.292	18.932	19.599	21.015	22.550
15	17.293	17.932	18.599	19.296	20.024	20.784	21.579	23.276	25.129
16	18.639	19.380	20.157	20.971	21.825	22.719	23.657	25.673	27.888
17	20.012	20.865	21.762	22.705	23.698	24.742	25.840	28.213	30.840
18	21.412	22.386	23.414	24.500	25.645	26.855	28.132	30.906	33.999
19	22.841	23.946	25.117	26.357	27.671	29.064	30.539	33.760	37.379
20	24.297	25.545	26.870	28.280	29.778	31.371	33.066	36.786	40.995
25	32.030	34.158	36.459	38.950	41.646	44.565	47.727	54.865	63.249
30	40.568	43.903	47.575	51.623	56.085	61.007	66.439	79.058	94.461
40	60.402	67.403	75.401	84.550	95.026	107.03	120.80	154.76	199.64
50	84.579	97.484	112.80	131.00	152.67	178.50	209.35	290.34	406.53
60	114.05	135.99	163.05	196.52	237.99	289.50	353.58	533.13	813.52

Formula: $v = \{[1 + (r/100)]^n - 1\} \div (r/100)$
$= (z - 1) \div (r/100)$.

PRINCIPAL WHICH WILL AMOUNT TO A GIVEN SUM

The principal P, which, if placed at compound interest to-day, will amount to a give sum A at the end of n years is $P = A \times x'$ or $P = A \times y'$ or $P = A \times z'$, according the interest (at the rate of r per cent. per annum) is compounded annually, semi-annuall or quarterly; the factor x' or y' or z' being taken from the following tables.

Values of x'. (Interest compounded annually; $P = A \times x'$)

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	.98039	.97561	.97087	.96618	.96154	.95694	.95238	.94340	.93458
2	.96117	.95181	.94260	.93351	.92456	.91573	.90703	.89000	.87344
3	.94232	.92860	.91514	.90194	.88900	.87630	.86384	.83962	.81630
4	.92385	.90595	.88849	.87144	.85480	.83856	.82270	.79209	.76290
5	.90573	.88385	.86261	.84197	.82193	.80245	.78353	.74726	.71299
6	.88797	.86230	.83748	.81350	.79031	.76790	.74622	.70496	.66634
7	.87056	.84127	.81309	.78599	.75992	.73483	.71068	.66506	.62275
8	.85349	.82075	.78941	.75941	.73069	.70319	.67684	.62741	.58201
9	.83676	.80073	.76642	.73373	.70259	.67290	.64461	.59190	.54393
10	.82035	.78120	.74409	.70892	.67556	.64393	.61391	.55839	.50835
11	.80426	.76214	.72242	.68495	.64958	.61620	.58468	.52679	.47509
12	.78849	.74356	.70138	.66178	.62460	.58966	.55684	.49697	.44401
13	.77303	.72542	.68095	.63940	.60057	.56427	.53032	.46884	.41496
14	.75788	.70773	.66112	.61778	.57748	.53997	.50507	.44230	.38783
15	.74301	.69047	.64186	.59689	.55526	.51672	.48102	.41727	.36245
16	.72845	.67362	.62317	.57671	.53391	.49447	.45811	.39365	.33873
17	.71416	.65720	.60502	.55720	.51337	.47318	.43630	.37136	.31657
18	.70016	.64117	.58739	.53836	.49363	.45280	.41552	.35034	.29586
19	.68643	.62553	.57029	.52016	.47464	.43330	.39573	.33051	.27651
20	.67297	.61027	.55368	.50257	.45639	.41464	.37689	.31180	.25842
25	.60953	.53939	.47761	.42315	.37512	.33273	.29530	.23300	.18425
30	.55207	.47674	.41199	.35628	.30832	.26700	.23138	.17411	.13137
40	.45289	.37243	.30656	.25257	.20829	.17193	.14205	.09722	.06678
50	.37153	.29094	.22811	.17905	.14071	.11071	.08720	.05429	.03395
60	.30478	.22728	.16973	.12693	.09506	.07129	.05354	.03031	.01726

Formula: $x' = [1 + (r/100)]^{-n} = 1/x$.

Values of y'. (Interest compounded semi-annually; $P = A \times y'$)

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	.98030	.97546	.97066	.96590	.96117	.95647	.95181	.94260	.93351
2	.96098	.95152	.94218	.93296	.92385	.91484	.90595	.88849	.87144
3	.94205	.92817	.91454	.90114	.88797	.87502	.86230	.83748	.81350
4	.92348	.90540	.88771	.87041	.85349	.83694	.82075	.78941	.75941
5	.90529	.88318	.86167	.84073	.82035	.80051	.78120	.74409	.70892
6	.88745	.86151	.83639	.81206	.78849	.76567	.74356	.70138	.66178
7	.86996	.84037	.81185	.78436	.75788	.73234	.70773	.66112	.61778
8	.85282	.81975	.78803	.75762	.72845	.70047	.67362	.62317	.57671
9	.83602	.79963	.76491	.73178	.70016	.66998	.64117	.58739	.53836
10	.81954	.78001	.74247	.70682	.67297	.64082	.61027	.55368	.50257
11	.80340	.76087	.72069	.68272	.64684	.61292	.58086	.52189	.46915
12	.78757	.74220	.69954	.65944	.62172	.58625	.55288	.49193	.43796
13	.77205	.72398	.67902	.63695	.59758	.56073	.52623	.46369	.40884
14	.75684	.70622	.65910	.61523	.57437	.53632	.50088	.43708	.38165
15	.74192	.68889	.63976	.59425	.55207	.51298	.47674	.41199	.35628
16	.72730	.67198	.62099	.57398	.53063	.49065	.45377	.38834	.33259
17	.71297	.65549	.60277	.55441	.51003	.46930	.43191	.36604	.31048
18	.69892	.63941	.58509	.53550	.49022	.44887	.41109	.34503	.28983
19	.68515	.62372	.56792	.51724	.47119	.42933	.39128	.32523	.27056
20	.67165	.60841	.55126	.49960	.45289	.41065	.37243	.30656	.25257
25	.60804	.53734	.47500	.42003	.37153	.32873	.29094	.22811	.17905
30	.55045	.47457	.40930	.35313	.30478	.26315	.22728	.16973	.12693
40	.45112	.37017	.30389	.24960	.20511	.16863	.13870	.09398	.06379
50	.36971	.28873	.22563	.17642	.13803	.10806	.08465	.05203	.03206
60	.30299	.22521	.16752	.12470	.09289	.06925	.05166	.02881	.01611

Formula: $y' = [1 + (r/200)]^{-2n} = 1/y$.

Values of z'. (Interest compounded quarterly; $P = A \times z'$; see opposite page)

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	.98025	.97539	.97055	.96575	.96098	.95624	.95152	.94218	.93296
2	.96089	.95138	.94198	.93268	.92348	.91439	.90540	.88771	.87041
3	.94191	.92796	.91424	.90074	.88745	.87437	.86151	.83639	.81206
4	.92330	.90512	.88732	.86989	.85282	.83611	.81975	.78803	.75762
5	.90506	.88284	.86119	.84010	.81954	.79952	.78001	.74247	.70682
6	.88719	.86111	.83583	.81132	.78757	.76453	.74220	.69954	.65944
7	.86966	.83991	.81122	.78354	.75684	.73107	.70622	.65910	.61523
8	.85248	.81924	.78733	.75670	.72730	.69908	.67198	.62099	.57390
9	.83564	.79908	.76415	.73079	.69892	.66849	.63941	.58509	.53550
10	.81914	.77941	.74165	.70576	.67165	.63923	.60841	.55126	.49960
11	.80296	.76022	.71981	.68159	.64545	.61126	.57892	.51939	.46611
12	.78710	.74151	.69861	.65825	.62026	.58451	.55086	.48936	.43486
13	.77155	.72326	.67804	.63570	.59606	.55893	.52415	.46107	.40570
14	.75631	.70546	.65808	.61393	.57280	.53447	.49874	.43441	.37851
15	.74137	.68809	.63870	.59291	.55045	.51108	.47457	.40930	.35313
16	.72673	.67115	.61989	.57260	.52897	.48871	.45156	.38563	.32946
17	.71237	.65464	.60164	.55299	.50833	.46733	.42967	.36334	.30737
18	.69830	.63852	.58392	.53405	.48850	.44687	.40884	.34233	.28676
19	.68451	.62281	.56673	.51576	.46944	.42732	.38903	.32254	.26754
20	.67099	.60748	.55004	.49810	.45112	.40862	.37017	.30389	.24960
25	.60729	.53630	.47369	.41845	.36971	.32670	.28873	.22563	.17642
30	.54963	.47347	.40794	.35154	.30299	.26120	.22521	.16752	.12470
40	.45023	.36903	.30255	.24810	.20351	.16697	.13702	.09235	.06230
50	.36880	.28762	.22438	.17510	.13669	.10673	.08337	.05091	.03113
60	.30210	.22417	.16641	.12358	.09181	.06823	.05072	.02806	.01555

Formula: $z' = [1 + (r/400)]^{-4n} = 1/z$.

ANNUITY WHICH WILL AMOUNT TO A GIVEN SUM (SINKING FUND)

The annual payment, Y, which, if set aside at the end of each year, will amount with accumulated interest to a given sum S at the end of n years is $Y = S \times v'$, where the factor v' is given below. (Interest at r per cent. per annum, compounded annually.)

Values of v'

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
2	.49505	.49383	.49261	.49140	.49020	.48900	.48780	.48544	.48309
3	.32675	.32514	.32353	.32193	.32035	.31877	.31721	.31411	.31105
4	.24262	.24082	.23903	.23725	.23549	.23374	.23201	.22859	.22523
5	.19216	.19025	.18835	.18648	.18463	.18279	.18097	.17740	.17389
6	.15853	.15655	.15460	.15267	.15076	.14888	.14702	.14336	.13980
7	.13451	.13250	.13051	.12854	.12661	.12470	.12282	.11914	.11555
8	.11651	.11447	.11246	.11048	.10853	.10661	.10472	.10104	.09747
9	.10252	.10046	.09843	.09645	.09449	.09257	.09069	.08702	.08349
10	.09133	.08926	.08723	.08524	.08329	.08138	.07950	.07587	.07238
11	.08218	.08011	.07808	.07609	.07415	.07225	.07039	.06679	.06336
12	.07456	.07249	.07046	.06848	.06655	.06467	.06283	.05928	.05590
13	.06812	.06605	.06403	.06206	.06014	.05828	.05646	.05296	.04965
14	.06260	.06054	.05853	.05657	.05467	.05282	.05102	.04758	.04434
15	.05783	.05577	.05377	.05183	.04994	.04811	.04634	.04296	.03979
16	.05365	.05160	.04961	.04768	.04582	.04402	.04227	.03895	.03586
17	.04997	.04793	.04595	.04404	.04220	.04042	.03870	.03544	.03243
18	.04670	.04467	.04271	.04082	.03899	.03724	.03555	.03236	.02941
19	.04378	.04176	.03981	.03794	.03614	.03441	.03275	.02962	.02675
20	.04116	.03915	.03722	.03536	.03358	.03188	.03024	.02718	.02439
25	.03122	.02928	.02743	.02567	.02401	.02244	.02095	.01823	.01581
30	.02465	.02278	.02102	.01937	.01783	.01639	.01505	.01265	.01059
40	.01656	.01484	.01326	.01183	.01052	.00934	.00828	.00646	.00467
50	.01182	.01026	.00887	.00763	.00655	.00560	.00478	.00344	.00238
60	.00877	.00735	.00613	.00509	.00420	.00345	.00283	.00188	.00121

Formula: $v' = (r/100) \div [[1 + (r/100)]^n - 1] = 1/v$.

PRESENT WORTH OF AN ANNUITY

The capital C, which, if placed at interest to-day, will provide for a given annual payment Y for a term of n years before it is exhausted is $C = Y \times w$, where the factor w is given below. (Interest at r per cent. per annum, compounded annually.)

Values of w

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
1	0.9804	0.9756	0.9709	0.9662	0.9615	0.9569	0.9524	0.9434	0.9346
2	1.9416	1.9274	1.9135	1.8997	1.8861	1.8727	1.8594	1.8334	1.8080
3	2.8839	2.8560	2.8286	2.8016	2.7751	2.7490	2.7232	2.6730	2.6243
4	3.8077	3.7620	3.7171	3.6731	3.6299	3.5875	3.5460	3.4651	3.3872
5	4.7135	4.6458	4.5797	4.5151	4.4518	4.3900	4.3295	4.2124	4.1002
6	5.6014	5.5081	5.4172	5.3286	5.2421	5.1579	5.0757	4.9173	4.7665
7	6.4720	6.3494	6.2303	6.1145	6.0021	5.8927	5.7864	5.5824	5.3893
8	7.3255	7.1701	7.0197	6.8740	6.7327	6.5959	6.4632	6.2098	5.9713
9	8.1622	7.9709	7.7861	7.6077	7.4353	7.2688	7.1078	6.8017	6.5152
10	8.9826	8.7521	8.5302	8.3166	8.1109	7.9127	7.7217	7.3601	7.0236
11	9.7868	9.5142	9.2526	9.0016	8.7605	8.5289	8.3064	7.8869	7.4987
12	10.575	10.258	9.9540	9.6633	9.3851	9.1186	8.8633	8.3838	7.9427
13	11.348	10.983	10.635	10.303	9.9856	9.6829	9.3936	8.8527	8.3577
14	12.106	11.691	11.296	10.921	10.563	10.223	9.8986	9.2950	8.7455
15	12.849	12.381	11.938	11.517	11.118	10.740	10.380	9.7122	9.1079
16	13.578	13.055	12.561	12.094	11.652	11.234	10.838	10.106	9.4466
17	14.292	13.712	13.166	12.651	12.166	11.707	11.274	10.477	9.7632
18	14.992	14.353	13.754	13.190	12.659	12.160	11.690	10.828	10.059
19	15.678	14.979	14.324	13.710	13.134	12.593	12.085	11.158	10.336
20	16.351	15.589	14.877	14.212	13.590	13.008	12.462	11.470	10.594
25	19.523	18.424	17.413	16.482	15.622	14.828	14.094	12.783	11.654
30	22.396	20.930	19.600	18.392	17.292	16.289	15.372	13.765	12.409
40	27.355	25.103	23.115	21.355	19.793	18.402	17.159	15.046	13.332
50	31.424	28.362	25.730	23.456	21.482	19.762	18.256	15.762	13.801
60	34.761	30.909	27.676	24.945	22.623	20.638	18.929	16.161	14.039

Formula: $w = [1 - [1 + (r/100)]^{-n}] \div [r/100] = v/x.$

ANNUITY PROVIDED FOR BY A GIVEN CAPITAL

The annual payment Y provided for for a term of n years by a given capital C placed at interest to-day is $Y = C \times w'$. (Interest at r per cent. per annum, compounded annually; the fund supposed to be exhausted at the end of the term.)

Values of w'

Years	$r = 2$	2½	3	3½	4	4½	5	6	7
2	.51505	.51883	.52261	.52640	.53020	.53400	.53780	.54544	.55309
3	.34675	.35014	.35353	.35693	.36035	.36377	.36721	.37411	.38105
4	.26262	.26582	.26903	.27225	.27549	.27874	.28201	.28859	.29523
5	.21216	.21525	.21835	.22148	.22463	.22779	.23097	.23740	.24389
6	.17853	.18155	.18460	.18767	.19076	.19388	.19702	.20336	.20980
7	.15451	.15750	.16051	.16354	.16661	.16970	.17282	.17914	.18555
8	.13651	.13947	.14246	.14548	.14853	.15161	.15472	.16104	.16747
9	.12252	.12546	.12843	.13145	.13449	.13757	.14069	.14702	.15349
10	.11133	.11426	.11723	.12024	.12329	.12638	.12950	.13587	.14238
11	.10218	.10511	.10808	.11109	.11415	.11725	.12039	.12679	.13336
12	.09456	.09749	.10046	.10348	.10655	.10967	.11283	.11928	.12590
13	.08812	.09105	.09403	.09706	.10014	.10328	.10646	.11296	.11965
14	.08260	.08554	.08853	.09157	.09467	.09782	.10102	.10758	.11434
15	.07783	.08077	.08377	.08683	.08994	.09311	.09634	.10296	.10979
16	.07365	.07660	.07961	.08268	.08582	.08902	.09227	.09895	.10586
17	.06997	.07293	.07595	.07904	.08220	.08542	.08870	.09544	.10243
18	.06670	.06967	.07271	.07582	.07899	.08224	.08555	.09236	.09941
19	.06378	.06676	.06981	.07294	.07614	.07941	.08275	.08962	.09675
20	.06116	.06415	.06722	.07036	.07358	.07688	.08024	.08718	.09439
25	.05122	.05428	.05743	.06067	.06401	.06744	.07095	.07823	.08581
30	.04465	.04778	.05102	.05437	.05783	.06139	.06505	.07265	.08059
40	.03656	.03984	.04326	.04683	.05052	.05434	.05828	.06646	.07467
50	.03182	.03526	.03887	.04263	.04655	.05060	.05478	.06344	.07238
60	.02877	.03235	.03613	.04009	.04420	.04845	.05283	.06188	.07121

Formula: $w' = [r/100] \div [1 - [1 + (r/100)]^{-n}] = 1/w = v' + (r/100).$

DECIMAL EQUIVALENTS

From minutes and seconds into decimal parts of a degree

0′	0°.0000	0″	0°.0000
1	.0167	1	.0003
2	.0333	2	.0006
3	.05	3	.0008
4	.0667	4	.0011
5′	.0833	5″	.0014
6	.10	6	.0017
7	.1167	7	.0019
8	.1333	8	.0022
9	.15	9	.0025
10′	0°.1667	10″	0°.0028
1	.1833	1	.0031
2	.20	2	.0033
3	.2167	3	.0036
4	.2333	4	.0039
15′	.25	15″	.0042
6	.2667	6	.0044
7	.2833	7	.0047
8	.30	8	.005
9	.3167	9	.0053
20′	0°.3333	20″	0°.0056
1	.35	1	.0058
2	.3667	2	.0061
3	.3833	3	.0064
4	.40	4	.0067
25′	.4167	25″	.0069
6	.4333	6	.0072
7	.45	7	.0075
8	.4667	8	.0078
9	.4833	9	.0081
30′	0°.50	30″	0°.0083
1	.5167	1	.0086
2	.5333	2	.0089
3	.55	3	.0092
4	.5667	4	.0094
35′	.5833	35″	.0097
6	.60	6	.01
7	.6167	7	.0103
8	.6333	8	.0106
9	.65	9	.0108
40′	0°.6667	40″	0°.0111
1	.6833	1	.0114
2	.70	2	.0117
3	.7167	3	.0119
4	.7333	4	.0122
45′	.75	45″	.0125
6	.7667	6	.0128
7	.7833	7	.0131
8	.80	8	.0133
9	.8167	9	.0136
50′	0°.8333	50″	0°.0139
1	.85	1	.0142
2	.8667	2	.0144
3	.8833	3	.0147
4	.90	4	.015
55′	.9167	55″	.0153
6	.9333	6	.0156
7	.95	7	.0158
8	.9667	8	.0161
9	.9833	9	.0164
60′	1.00	60″	0°.0167

From decimal parts of a degree into minutes and seconds (exact values)

0°.00	0′	0°.50	30′
1	0′ 36″	1	30′ 36″
2	1′ 12″	2	31′ 12″
3	1′ 48″	3	31′ 48″
4	2′ 24″	4	32′ 24″
0°.05	3′	0°.55	33′
6	3′ 36″	6	33′ 36″
7	4′ 12″	7	34′ 12″
8	4′ 48″	8	34′ 48″
9	5′ 24″	9	35′ 24″
0°.10	6′	0°.60	36′
1	6′ 36″	1	36′ 36″
2	7′ 12″	2	37′ 12″
3	7′ 48″	3	37′ 48″
4	8′ 24″	4	38′ 24″
0°.15	9′	0°.65	39′
6	9′ 36″	6	39′ 36″
7	10′ 12″	7	40′ 12″
8	10′ 48″	8	40′ 48″
9	11′ 24″	9	41′ 24″
0°.20	12′	0°.70	42′
1	12′ 36″	1	42′ 36″
2	13′ 12″	2	43′ 12″
3	13′ 48″	3	43′ 48″
4	14′ 24″	4	44′ 24″
0°.25	15′	0°.75	45′
6	15′ 36″	6	45′ 36″
7	16′ 12″	7	46′ 12″
8	16′ 48″	8	46′ 48″
9	17′ 24″	9	47′ 24″
0°.30	18′	0°.80	48′
1	18′ 36″	1	48′ 36″
2	19′ 12″	2	49′ 12″
3	19′ 48″	3	49′ 48″
4	20′ 24″	4	50′ 24″
0°.35	21′	0°.85	51′
6	21′ 36″	6	51′ 36″
7	22′ 12″	7	52′ 12″
8	22′ 48″	8	52′ 48″
9	23′ 24″	9	53′ 24″
0°.40	24′	0°.90	54′
1	24′ 36″	1	54′ 36″
2	25′ 12″	2	55′ 12″
3	25′ 48″	3	55′ 48″
4	26′ 24″	4	56′ 24″
0°.45	27′	0°.95	57′
6	27′ 36″	6	57′ 36″
7	28′ 12″	7	58′ 12″
8	28′ 48″	8	58′ 48″
9	29′ 24″	9	59′ 24″
0°.50	30′	1°.00	60′

0°.000	0″.0
1	3″.6
2	7″.2
3	10″.8
4	14″.4
0°.005	18″
6	21″.6
7	25″.2
8	28″.8
9	32″.4
0°.010	36″

Common fractions

8 ths	16 ths	32 nds	64 ths	Exact decimal values
			1	.01 5625
		1	2	.03 125
			3	.04 6875
	1	2	4	.06 25
			5	.07 8125
		3	6	.09 375
			7	.10 9375
1	2	4	8	.12 5
			9	.14 0625
		5	10	.15 625
			11	.17 1875
	3	6	12	.18 75
			13	.20 3125
		7	14	.21 875
			15	.23 4375
2	4	8	16	.25
			17	.26 5625
		9	18	.28 125
			19	.29 6875
	5	10	20	.31 25
			21	.32 8125
		11	22	.34 375
			23	.35 9375
3	6	12	24	.37 5
			25	.39 0625
		13	26	.40 625
			27	.42 1875
	7	14	28	.43 75
			29	.45 3125
		15	30	.46 875
			31	.48 4375
4	8	16	32	.50
			33	.51 5625
		17	34	.53 125
			35	.54 6875
	9	18	36	.56 25
			37	.57 8125
		19	38	.59 375
			39	.60 9375
5	10	20	40	.62 5
			41	.64 0625
		21	42	.65 625
			43	.67 1875
	11	22	44	.68 75
			45	.70 3125
		23	46	.71 875
			47	.73 4375
6	12	24	48	.75
			49	.76 5625
		25	50	.78 125
			51	.79 6875
	13	26	52	.81 25
			53	.82 8125
		27	54	.84 375
			55	.85 9375
7	14	28	56	.87 5
			57	.89 0625
		29	58	.90 625
			59	.92 1875
	15	30	60	.93 75
			61	.95 3125
		31	62	.96 875
			63	.98 4375

WEIGHTS AND MEASURES

BY

LOUIS A. FISCHER

In the United States the measures of weight and length commonly employed are identical with the corresponding English units, but the capacity measures differ from those now in use in the British Empire, the U. S. gallon being defined as 231 cu. in. and the bushel as 2150.42 cu. in., whereas the corresponding British imperial units are, respectively, 277.418 cu. in., and 2219.344 cu. in. (1 imp. gal. = 1.2 U. S. gal., approx.; 1 imp. bu. = 1.03 U. S. bu., approx.).

The metric system of weights and measures was legalized and its use made permissive in the United States by an Act of Congress, passed in 1866. In 1872, by the concurrent action of the principal governments of the world, it was agreed to establish an International Bureau of Weights and Measures near Paris.

Prior to 1891 the British imperial yard was regarded as the real standard of the United States. In 1891, the Office of Weights and Measures (now Bureau of Standards) fixed the value of the United States yard in terms of the international meter, according to the ratio: one yard = 3600/3937 meters. At the same time, the pound was fixed in terms of the international kilogram, according to the relation: one pound = 453.59243 grams.

U. S. Customary Weights and Measures

Measures of Length

12 inches	= 1 foot
3 feet	= 1 yard
5½ yards = 16½ feet	= 1 rod, pole or perch
40 poles = 220 yards	= 1 furlong
8 furlongs = 1760 yards = 5280 feet	= 1 mile
3 miles	= 1 league
4 inches	= 1 hand
9 inches	= 1 span

Nautical Units

6080.2 feet	= 1 nautical mile
6 feet	= 1 fathom
120 fathoms	= 1 cable length
1 nautical mile per hr.	= 1 knot

Surveyor's or Gunter's Measure

7.92 inches	= 1 link
100 links = 66 ft. = 4 rods	= 1 chain
80 chains	= 1 mile
33⅓ inches	= 1 vara (Texas)

Measures of Area

144 square inches	= 1 square foot
9 square feet	= 1 square yard
30¼ square yards	= 1 square rod, pole or perch
160 square rods = 10 square chains = 43,560 sq. ft. = 5645 sq. varas (Texas)	= 1 acre
640 acres = 1 square mile	= 1 "section" of U. S. Govt. surveyed land
1 circular inch = area of circle 1 inch in diameter	= 0.7854 sq. in.
1 square inch	= 1.2732 cir. in.
1 circular mil	= area of circle 0.001 in. in diam.
1,000,000 cir. mils	= 1 cir. in.

Measures of Volume

1728 cubic inches	= 1 cubic foot
27 cubic feet	= 1 cubic yard
1 cord of wood	= 128 cu. ft.
1 perch of masonry	= 16½ to 25 cu. ft.

U. S. Customary Weights and Measures—(*continued*)

Measures of Volume

Liquid or Fluid Measure

4 gills	= 1 pint
2 pints	= 1 quart
4 quarts	= 1 gallon
7.4805 gallons	= 1 cubic foot

(There is no standard liquid "barrel.")

Apothecaries' Liquid Measure

60 minims = 1 liquid dram or drachm
8 drams = 1 liquid ounce
16 ounces = 1 pint

Water Measure

The **Miner's Inch** is the quantity of water that will pass through an orifice 1 sq. in. in cross-section under a head of from 4 to 6½ in., as fixed by statutes, and varies from 1/40 cu. ft. to 1/50 cu. ft. per sec. The units now most in use are 1 cu. ft. per sec. and 1 gal. per sec., the U. S. Reclamation Service employing the former. See p. 260.

Dry Measure

2 pints = 1 quart
8 quarts = 1 peck
4 pecks = 1 bushel

Shipping Measure

1 Register ton	= 100 cu. ft.
1 U. S. shipping ton	= 40 cu. ft.
	= 32.14 U. S. bu. / 31.14 imp. bu.
1 British shipping ton	= 42 cu. ft.
	= 32.70 imp. bu. / 33.75 U. S. bu.

Board Measure

1 board foot = 144 cu. in. = volume of board 1 ft. sq. and 1 in. thick.

No. of board feet in a log $= [\frac{1}{4}(d-4)]^2L$, where d = diam. of log (usually taken inside the bark at small end), in., and L = length of log, ft. The 4 in. deducted are an allowance for slab. This rule is variously known as the Doyle, Conn. River, St. Croix, Thurber, Moore and Beeman, and the Scribner rule.

Weights
(The grain is the same in all systems)

Avoirdupois Weight

16 drams = 437.5 grains	= 1 ounce
16 ounces = 7000 grains	= 1 pound
100 pounds	= 1 cental
2000 pounds	= 1 short ton
2240 pounds	= 1 long ton

Also (in Great Britain):

14 pounds	= 1 stone
2 stone = 28 lb.	= 1 quarter
4 quarters = 112 lb.	= 1 hundredweight (cwt.)
20 hundredweight	= 1 long ton

Troy Weight

24 grains	= 1 pennyweight (dwt.)
20 pennyweights = 480 grains	= 1 ounce
12 ounces = 5760 grains	= 1 pound

1 **Assay Ton** = 29,167 milligrams, or as many milligrams as there are troy ounces in a ton of 2000 lb. avoirdupois. Consequently, the number of milligrams of precious metal yielded by an assay ton of ore gives directly the number of troy ounces that would be obtained from a ton of 2000 lb. avoirdupois.

Apothecaries' Weight

20 grains	= 1 scruple ℈
3 scruples = 60 grains	= 1 dram ʒ
8 drams	= 1 ounce ℥
12 ounces = 5760 grains	= 1 pound

Weight for Precious Stones

1 carat = 200 milligrams
(Adopted by practically all important nations.)

Circular Measure

60 seconds	= 1 minute
60 minutes	= 1 degree
90 degrees	= 1 quadrant
360 degrees	= circumference
57.2957795 degrees (= 57° 17′44.806″)	= 1 radian (or angle having arc of length equal to radius)

METRIC SYSTEM

The fundamental unit of the metric system is the **meter**—the unit of length, from which the units of volume (**liter**) and of mass (**gram**) are derived. All other units are the decimal subdivisions or multiples of these. These three units are simply related: one cubic decimeter equals one liter, and one liter of water weighs one kilogram. The metric tables are formed by combining the words "meter," "gram," and "liter" with numerical prefixes.

All lengths, areas, and cubic measures in the following conversion tables are derived from the international meter. The customary weights are likewise derived from the kilogram. All capacities are based on the practical equivalent: 1 cubic decimeter equals 1 liter. (The liter is defined as the volume occupied by the mass of 1 kilogram of water under a pressure of 76 cm. of mercury and at the temperature of 4 deg. cent. According to the best information, 1 liter = 1.000027 cubic decimeters.)

The customary weights derived from the international kilogram are based on the value 1 avoirdupois lb. = 453.59243 grams. The value of the troy lb. is based on the same relation and also the equivalent 5760/7000 avoirdupois lb. equals 1 troy lb.

Metric Measures

Length			Area		
Unit	Symbol	Value in meters	Unit	Symbol	Value in sq. meters
Micron	μ	0.000001	....	.....	....
Millimeter	mm.	0.001	Sq. millimeter	$mm.^2$	0.000001
Centimeter	cm.	0.01	Sq. centimeter	$cm.^2$	0.0001
Decimeter	dm.	0.1	Sq. decimeter	$dm.^2$	0.01
Meter (unit)	m.	1.0	Sq. meter (centiare)	$m.^2$	1.0
Dekameter	dkm.	10.0	Sq. dekameter (are)	a.	100.0
Hectometer	hm.	100.0	Hectare	ha.	10,000.0
Kilometer	km.	1,000.0	Sq. kilometer	$km.^2$	1,000,000.0
Myriameter	Mm.	10,000.0	....	.....	....
Megameter	.....	1,000,000.0	....	.....	....

Volume			Cubic measure		
Unit	Symbol	Value in liters	Unit	Symbol	Value in cubic meters
Milliliter	ml. or $cm.^3$	0.001	Cubic kilometer	$km.^3$	10^9
Liter (unit)	l. or $dm.^3$	1.0	Cubic hectometer	$hm.^3$	10^6
Kiloliter	kl. or $m.^3$	1,000.0	Cubic dekameter	$dkm.^3$	10^3
Also					
Centiliter	cl.	0.01	Cubic meter	$m.^3$	1
Deciliter	dl.	0.1	Cubic decimeter	$dm.^3$	10^{-3}
Dekaliter	dkl.	10.0	Cubic centimeter	$cm.^3$	10^{-6}
Hectoliter	hl.	100.0	Cubic millimeter	$mm.^3$	10^{-9}
....	....	....	Cubic micron	μ^3	10^{-18}

Weight					
Unit	Symbol	Value in grams	Unit	Symbol	Value in grams
Microgram	....	0.000001	Dekagram	dkg.	10.0
Milligram	mg.	0.001	Hectogram	hg.	100.0
Centigram	cg.	0.01	Kilogram	kg.	1,000.0
Decigram	dg.	0.1	Myriagram	Mg.	10,000.0
Gram (unit)	g.	1.0	Quintal	q.	100,000.0
....	....	....	Ton	t.	1,000,000.0

SYSTEMS OF UNITS

The principal units of interest to mechanical engineers can all be derived from the three fundamental units of **force, length,** and **time.** These three fundamental units may be chosen at pleasure; each such choice gives rise to a "system" of units. The following table gives the units of the four "systems" most often met with in the literature.

The precise definitions of the fundamental units in these systems are as follows. (In these definitions the "standard pound body" and the "standard kilogram body" refer to two special lumps of metal, carefully preserved at London and Paris, respectively; the "standard locality" means sea level, 45 deg. latitude; or, more strictly, any locality in which the acceleration due to gravity has the value 980.665 cm. per sec.^{2} = 32.1740 ft. per sec.^{2}, which may be called the **standard acceleration.**

The **pound** (*force*) is the force required to support the standard pound body against gravity, *in vacuo*, in the standard locality; or, it is the force which, if applied to the standard pound body, supposed free to move, would give that body the "standard acceleration." The word "pound" is used for the unit of both force and mass, and consequently is ambiguous. To avoid uncertainty it is desirable to call the units "pound force" and "pound mass," respectively.

The **kilogram** (*force*) is the force required to support the standard kilogram against gravity, *in vacuo*, in the standard locality; or, it is the force which, if applied to the standard kilogram body, supposed free to move, would give that body the "standard acceleration." The word "kilogram" is used for the unit of both force and mass and consequently is ambiguous. To avoid uncertainty it is desirable to call the units "kilogram force" and "kilogram mass," respectively.

The **poundal** is the force which, if applied to the standard pound body, would give that body an acceleration of 1 ft. per sec.^{2}; that is, 1 poundal = 1/32.1740 of a pound force.

The **dyne** is the force which, if applied to the standard gram body, would give that body an acceleration of 1 cm. per sec.^{2}; that is, 1 dyne = 1/980.665 of a gram force.

Systems of Units

Name of unit	Dimensions of units in terms of F, L, T	British "gravitational" system, or "foot-pound-second" system	Metric "gravitational" system, or "kilogram-meter-second" system	Metric "absolute" system, or "C. G. S." system	British "absolute" system (little used)
Force	F	1 lb.	1 kg.	1 dyne	1 poundal
Length	L	1 ft.	1 m.	1 cm.	1 ft.
Time	T	1 sec.	1 sec.	1 sec.	1 sec.
Velocity	L/T	1 ft. per sec.	1 m. per sec.	1 cm. per sec.	1 ft. per sec.
Acceleration	L/T^2	1 ft. per sec.^{2}	1 m. per sec.^{2}	1 cm. per sec.^{2}	1 ft. per sec.^{2}
Pressure	F/L^2	1 lb. per ft.^{2}	1 kg. per m.^{2}	1 dyne per cm.^{2}	1 pdl. per ft.^{2}
Impulse or momentum	FT	1 lb.-sec.	1 kg.-sec.	1 dyne-sec.	1 pdl.-sec.
Work or energy	FL	1 ft.-lb.	1 kg.-m.	1 dyne-cm. = 1 "erg."	1 ft.-pdl.
Power	FL/T	1 ft.-lb. per sec.	1 kg.-m. per sec.	1 dyne-cm. per sec.	1 ft.-pdl. per sec.
Mass	$F/(L/T^2)$	1 lb. per (ft. per sec.^{2}) = 1 "slug."	1 kg. per (m. per sec.^{2}) = 1 "metric slug."	1 dyne per (cm. per sec.^{2}) = 1 gram mass.	1 pdl. per (ft. per sec.^{2}) = 1 pound mass.

NOTE. The "slug" (also called the "geepound," or the "engineer's unit of mass"), the "metric slug," and the "poundal" are never used in practice.

Other common units are as follows:

Work: 1 joule = 10^7 ergs = 10,000,000 dyne-cm.
1 kilowatt-hour = 3,600,000 joules = 3600 × 10^{10} dyne-cm.

Power: 1 horse power = 550 ft.-lb. per sec.
1 poncelet = 100 kg.-m. per sec.
1 force de cheval = 75 kg.-m. per sec.
1 watt = 1 joule per sec. = 10,000,000 dyne-cm. per sec.
1 kilowatt = 1000 watts = 10^{10} dyne-cm. per sec.

A new horse power of 550.220 ft.-lb. per sec., or 746 watts, has been proposed, but has not been accepted by mechanical engineers.

The *weight* of a body (in a given locality) always means a *force*, namely, the force, re-

quired to support the body against gravity (in that locality). When no particular locality is specified, the standard locality may be assumed. Thus, the "standard weight" of the pound body is 1 lb.; the "standard weight" of the kilogram body is 1 kg.

Heat Units. The units of heat commonly used are (1) the quantity of heat required to raise the temperature of 1 gram of water 1 deg. cent. at a mean temperature of 15 deg. cent., or (2) the heat required to raise the temperature of 1 lb. of water 1 deg. fahr. The former quantity is called the **gram-calorie** (small calorie), while the latter is known as the **British thermal unit** or B.t.u. The **kilogram-calorie** (large calorie), which is equal to 1000 g.-cal., is largely used in engineering work in metric countries.* 1 therm = 1 g.-cal.

Force Equivalents

Dynes × 10^6	Kilograms	Pounds	Poundals
1	**1.020** 0.00848	**2.248** 0.03518	**72.33** 1.85933
0.9807 $\bar{1}.99149$	**1**	**2.205** 0.34334	**70.93** 1.85084
0.4448 $\bar{1}.64819$	**0.4536** $\bar{1}.65667$	**1**	**32.17** 1.50750
0.01383 $\bar{2}.14067$	**0.01410** $\bar{2}.14916$	**0.03108** $\bar{2}.49249$	**1**

CONVERSION TABLES

Length Equivalents

Centimeters	Inches	Feet	Yards	Meters	Chains	Kilometers	Miles
1	**0.3937** $\bar{1}.59517$	**0.03281** $\bar{2}.51598$	**0.01094** $\bar{2}.03886$	**0.01** $\bar{2}.00000$	**0.0_34971** $\bar{4}.69644$	**10^{-5}** $\bar{5}.00000$	**0.0_56214** $\bar{6}.79335$
2.540 0.40483	**1**	**0.0_38333** $\bar{4}.92082$	**0.02778** $\bar{2}.44370$	**0.0254** $\bar{2}.40483$	**0.0_41263** $\bar{5}.10127$	**0.0_4254** $\bar{5}.40483$	**0.0_41578** $\bar{5}.19818$
30.48 1.48402	**12** 1.07918	**1**	**0.3333** $\bar{1}.52288$	**0.3048** $\bar{1}.48402$	**0.01515** $\bar{2}.18046$	**0.0_33098** $\bar{4}.48402$	**0.0_31645** $\bar{4}.21608$
91.14 1.96114	**36** 1.55630	**3** 0.47712	**1**	**0.9144** $\bar{1}.96114$	**0.04545** $\bar{2}.65758$	**0.0_39144** $\bar{4}.96114$	**0.0_35682** $\bar{4}.75449$
100 2.00000	**39.37** 1.59517	**3.281** 0.51598	**1.0936** 0.03886	**1**	**0.04971** $\bar{2}.69644$	**0.001** $\bar{3}.00000$	**0.0_36214** $\bar{4}.79335$
2012 3.30356	**792** 2.89873	**66** 1.81954	**22** 1.34242	**20.12** 1.30356	**1**	**0.02012** $\bar{2}.30356$	**0.0125** $\bar{2}.09691$
100000 5.00000	**39370** 4.59517	**3281** 3.51598	**1093.6** 3.03886	**1000** 3.00000	**49.71** 1.69644	**1**	**0.6214** $\bar{1}.79335$
160925 5.20665	**63360** 4.80182	**5280** 3.72263	**1760** 3.24551	**1609** 3.20665	**80** 1.90309	**1.609** 0.20665	**1**

The equivalents are given in the heavier type. Logarithms of the equivalents are given immediately below.

Subscripts after any figure, 0_3, 9_4, etc., mean that that figure is to be repeated the indicated number of times.

Conversion of Lengths

	Inches to millimeters	Millimeters to inches	Feet to meters	Meters to feet	Yards to meters	Meters to yards	Miles to kilometers	Kilometers to miles
1	25.40	0.03937	0.3048	3.281	0.9144	1.094	1.609	0.6214
2	50.80	0.07874	0.6096	6.562	1.829	2.187	3.219	1.243
3	76.20	0.1181	0.9144	9.842	2.743	3.281	4.828	1.864
4	101.60	0.1575	1.219	13.12	3.658	4.374	6.437	2.485
5	127.00	0.1968	1.524	16.40	4.572	5.486	8.047	3.107
6	152.40	0.2362	1.829	19.68	5.486	6.562	9.656	3.728
7	177.80	0.2756	2.134	22.97	6.401	7.655	11.27	4.350
8	203.20	0.3150	2.438	26.25	7.315	8.749	12.87	4.971
9	228.60	0.3543	2.743	29.53	8.230	9.842	14.48	5.592

*See Marks' Mechanical Engineers' Handbook.

Mechanical Equivalent of Heat. See p. 311.* The value most commonly accepted among American engineers as the work equivalent of 1 **mean B.t.u.** is 777.5 ft.-lb., and the **mean gram-calorie** = 4.183 joules, which are the values used throughout this book. The U. S. Bureau of Standards does not recommend any special value; for its own purposes it takes the 59 deg. fahr. B.t.u. as 778.2 ft.-lb. and the 68 deg. B.t.u. as 777.5 ft.-lb. The 15 deg. calorie = 4.187 joules; 20 deg. calorie = 4.183 joules. There is an uncertainty of about 1 part in 1000 in these values.

Conversion of Lengths: Inches and Millimeters

Common fractions of an inch to millimeters
(From 1/64 to 1 in.)

64ths	Millimeters	64ths	Millimeters	64ths	Millimeters	64ths	Millimeters	64ths	Millimeters	64ths	Millimeters
1	0.397	13	5.159	25	9.922	37	14.684	49	19.447	57	22.622
2	0.794	14	5.556	26	10.319	38	15.081	50	19.844	58	23.019
3	1.191	15	5.953	27	10.716	39	15.478	51	20.241	59	23.416
4	1.588	16	6.350	28	11.113	40	15.875	52	20.638	60	23.813
5	1.984	17	6.747	29	11.509	41	16.272	53	21.034	61	24.209
6	2.381	18	7.144	30	11.906	42	16.669	54	21.431	62	24.606
7	2.778	19	7.541	31	12.303	43	17.066	55	21.828	63	25.003
8	3.175	20	7.938	32	12.700	44	17.463	56	22.225	64	25.400
9	3.572	21	8.334	33	13.097	45	17.859				
10	3.969	22	8.731	34	13.494	46	18.256				
11	4.366	23	9.128	35	13.891	47	18.653				
12	4.763	24	9.525	36	14.288	48	19.050				

Decimals of an inch to millimeters. (From 0.01 in. to 0.99 in.)

	0	1	2	3	4	5	6	7	8	9
.0		0.254	0.508	0.762	1.016	1.270	1.524	1.778	2.032	2.286
.1	2.540	2.794	3.048	3.302	3.556	3.810	4.064	4.318	4.572	4.826
.2	5.080	5.334	5.588	5.842	6.096	6.350	6.604	6.858	7.112	7.366
.3	7.620	7.874	8.128	8.382	8.636	8.890	9.144	9.398	9.652	9.906
.4	10.160	10.414	10.668	10.922	11.176	11.430	11.684	11.938	12.192	12.446
.5	12.700	12.954	13.208	13.462	13.716	13.970	14.224	14.478	14.732	14.986
.6	15.240	15.494	15.748	16.002	16.256	16.510	16.764	17.018	17.272	17.526
.7	17.780	18.034	18.288	18.542	18.796	19.050	19.304	19.558	19.812	20.066
.8	20.320	20.574	20.828	21.082	21.336	21.590	21.844	22.098	22.352	22.606
.9	22.860	23.114	23.368	23.622	23.876	24.130	24.384	24.638	24.892	25.146

Millimeters to decimals of an inch. (From 1 to 99 mm.)

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
0		0.0394	0.0787	0.1181	0.1575	0.1969	0.2362	0.2756	0.3150	0.3543
1	0.3937	0.4331	0.4724	0.5118	0.5512	0.5906	0.6299	0.6693	0.7087	0.7480
2	0.7874	0.8268	0.8661	0.9055	0.9449	0.9843	1.0236	1.0630	1.1024	1.1417
3	1.1811	1.2205	1.2598	1.2992	1.3386	1.3780	1.4173	1.4567	1.4961	1.5354
4	1.5748	1.6142	1.6535	1.6929	1.7323	1.7717	1.8110	1.8504	1.8898	1.9291
5	1.9685	2.0079	2.0472	2.0866	2.1260	2.1654	2.2047	2.2441	2.2835	2.3228
6	2.3622	2.4016	2.4409	2.4803	2.5197	2.5591	2.5984	2.6378	2.6772	2.7165
7	2.7559	2.7953	2.8346	2.8740	2.9134	2.9528	2.9921	3.0315	3.0709	3.1102
8	3.1496	3.1890	3.2283	3.2677	3.3071	3.3465	3.3858	3.4252	3.4646	3.5039
9	3.5433	3.5827	3.6220	3.6614	3.7008	3.7402	3.7795	3.8189	3.8583	3.8976

*See Marks' MECHANICAL ENGINEERS' HANDBOOK.

Area Equivalents

(For conversion table see p. 77)

Square meters	Square inches	Square feet	Square yards	Square rods	Square chains	Roods	Acres	Square miles or sections
1	**1550** 3.19033	**10.76** 1.03197	**1.196** 0.07773	**0.0395** $\bar{2}$.59699	**0.002471** $\bar{3}$.39288	**0.0_39884** $\bar{3}$.99494	**0.0_32471** $\bar{4}$.39288	**0.0_63861** $\bar{7}$.58670
0.0_36452 $\bar{4}$ 80967	**1**	**0.006944** $\bar{3}$.84164	**0.0011** $\bar{3}$.88740	**0.0_42551** $\bar{5}$.40667	**0.0_51594** $\bar{6}$.20255	**0.0_66377** $\bar{7}$.80461	**0.0_61594** $\bar{7}$.20255	**0.0_94910** $\overline{10}$.39637
0.09290 $\bar{2}$.96803	**144** 2.15836	**1**	**0.1111** $\bar{1}$.04576	**0.003673** $\bar{3}$.56503	**0.0_32296** $\bar{4}$.36091	**0.0_49184** $\bar{5}$.96297	**0.0_42296** $\bar{4}$.36091	**0.0_73587** $\bar{8}$.55473
0.8361 $\bar{1}$.92227	**1296** 3.11260	**9** 0.95424	**1**	**0.03306** $\bar{2}$.51927	**0.002066** $\bar{3}$.31515	**0.0_38264** $\bar{4}$.91721	**0.0002066** $\bar{4}$.31515	**0.0_63228** $\bar{7}$.50898
25.29 1.40300	**39204** 4.59333	**272.25** 2.43497	**30.25** 1.48072	**1**	**0.0625** $\bar{2}$.79588	**0.02500** $\bar{2}$.39794	**0.00625** $\bar{3}$.79588	**0.0_59766** $\bar{6}$.98970
404.7 2.60712	**627264** 5.79745	**4356** 3.63909	**484** 2.68484	**16** 1.20412	**1**	**0.4** $\bar{1}$.60206	**0.1** $\bar{1}$.00000	**0.0001562** $\bar{4}$.19382
1012 3.00506	**1568160** 6.19539	**10890** 4.03703	**1210** 3.08278	**40** 1.60206	**2.5** 0.39794	**1**	**0.25** $\bar{1}$.39794	**0.0_33906** $\bar{4}$.59176
4047 3.60712	**6272640** 6.79745	**43560** 4.63909	**4840** 3.68484	**160** 2.20412	**10** 1.00000	**4** 0.60206	**1**	**0.001562** $\bar{3}$.19382
2589_38 6.41330		**27878400** 7.44527	**3097600** 6.49102	**102400** 5.01030	**6400** 3.80618	**2560** 3.40824	**640** 2.80618	**1**

(1 hectare = 100 ares = 10,000 centiares or square meters)

Volume and Capacity Equivalents

(For conversion table see p. 77)

Cubic inches	Cubic feet	Cubic yards	U. S. Apothecary liquid ounces	U. S. quarts Liquid	U. S. quarts Dry	U. S. gallons Liquid	U. S. gallons Dry	Bushels U. S.	Liters (l)
1	**0.0_35787** $\bar{4}$.76246	**0.0_42143** $\bar{5}$.33109	**0.5541** $\bar{1}$.74360	**0.01732** $\bar{2}$.23845	**0.01488** $\bar{2}$.17263	**0.0_24329** $\bar{3}$.63639	**0.0_23720** $\bar{3}$.57057	**0.0_34650** $\bar{4}$.66748	**0.01639** $\bar{2}$.21450
1728 3.23754	**1**	**0.03704** $\bar{2}$.56864	**957.5** 2.98114	**29.92** 1.47599	**25.71** 1.41017	**7.481** 0.87393	**6.429** 0.80811	**0.8036** $\bar{1}$.90502	**28.32** 1.45205
46656 4.66891	**27** 1.43136	**1**	**25853** 4.41251	**807.9** 2.90736	**694.3** 2.84153	**202.0** 2.30530	**173.6** 2.23948	**21.70** 1.33638	**764.6** 2.88341
1.805 0.25640	**0.001044** $\bar{3}$.01886	**0.0_43868** $\bar{5}$.58749	**1**	**0.03125** $\bar{2}$.49485	**0.02686** $\bar{2}$.42903	**0.007813** $\bar{3}$.89279	**0.006714** $\bar{3}$.82697	**0.0_38392** $\bar{4}$.92388	**0.02957** $\bar{2}$.47091
57.75 1.76155	**0.03342** $\bar{2}$.52401	**0.001238** $\bar{3}$.09264	**32** 1.50515	**1**	**0.8594** $\bar{1}$.93418	**0.25** $\bar{1}$.39794	**0.2148** $\bar{1}$.33212	**0.02686** $\bar{2}$.42903	**0.9464** $\bar{1}$.97606
67.20 1.82737	**0.03889** $\bar{2}$.58983	**0.001440** $\bar{3}$.15847	**37.24** 1.57097	**1.164** 0.06582	**1**	**0.2909** $\bar{1}$.46376	**0.25** $\bar{1}$.39794	**0.03125** $\bar{2}$.49485	**1.101** 0.04188
231 2.36361	**0.1337** $\bar{1}$.12607	**0.004951** $\bar{3}$.69470	**128** 2.10721	**4** 0.60206	**3.437** 0.53624	**1**	**0.8594** $\bar{1}$.93418	**0.1074** $\bar{1}$.03109	**3.785** 0.57812
268.8 2.42943	**0.1556** $\bar{1}$.19189	**0.005761** $\bar{3}$.76053	**148.9** 2.17303	**4.655** 0.66788	**4** 0.60206	**1.164** 0.06582	**1**	**0.125** $\bar{1}$.09691	**4.405** 0.64394
2150 3.33252	**1.244** 0.09498	**0.04609** $\bar{2}$.66362	**1192** 3.07612	**37.24** 1.57097	**32** 1.50515	**9.309** 0.96891	**8** 0.90309	**1**	**35.24** 1.54703
61.02 1.78550	**0.03531** $\bar{2}$.54795	**0.001308** $\bar{3}$.11659	**33.81** 1.52909	**1.057** 0.02394	**0.9081** $\bar{1}$.95812	**0.2642** $\bar{1}$.42188	**0.2270** $\bar{1}$.35606	**0.02838** $\bar{2}$.45297	**1**

The equivalents are given in the heavier type. Logarithms of the equivalents are given immediately below.

Subscripts after any figure, 0_3, 9_4, etc., mean that that figure is to be repeated the indicated number of times.

Conversion of Areas

	Sq. in. to sq. cm.	Sq. cm. to sq. in.	Sq. ft. to sq. m.	Sq. m. to sq. ft.	Sq. yd. to sq. m.	Sq. m. to sq. yd.	Acres to hectares	Hectares to acres	Sq. mi. to sq. km.	Sq. km. to sq. mi.
1	6.452	0.1550	0.0929	10.76	0.8361	1.196	0.4047	2.471	2.590	0.3861
2	12.90	0.3100	0.1858	21.53	1.672	2.392	0.8094	4.942	5.180	0.7722
3	19.35	0.4650	0.2787	32.29	2.508	3.588	1.214	7.413	7.770	1.158
4	25.81	0.6200	0.3716	43.06	3.345	4.784	1.619	9.884	10.360	1.544
5	32.26	0.7750	0.4645	53.82	4.181	5.980	2.023	12.355	12.950	1.931
6	38.71	0.9300	0.5574	64.58	5.017	7.176	2.428	14.826	15.540	2.317
7	45.16	1.085	0.6503	75.35	5.853	8.372	2.833	17.297	18.130	2.703
8	51.61	1.240	0.7432	86.11	6.689	9.568	3.237	19.768	20.720	3.089
9	58.06	1.395	0.8361	96.87	7.525	10.764	3.642	22.239	23.310	3.475

Conversion of Volumes or Cubic Measure

	Cu. in. to cu. cm.	Cu. cm. to cu. in.	Cu. ft. to cu. m.	Cu. m. to cu. ft.	Cu. yd. to Cu. m.	Cu. m. to cu. yd.	Gallons to cu. ft.	Cu. ft. to gallons
1	16.39	0.06102	0.02832	35.31	0.7646	1.308	0.1337	7.481
2	32.77	0.1220	0.05663	70.63	1.529	2.616	0.2674	14.96
3	49.16	0.1831	0.08495	105.9	2.294	3.924	0.4011	22.44
4	65.55	0.2441	0.1133	141.3	3.058	5.232	0.5348	29.92
5	81.94	0.3051	0.1416	176.6	3.823	6.540	0.6685	37.41
6	98.32	0.3661	0.1699	211.9	4.587	7.848	0.8022	44.89
7	114.7	0.4272	0.1982	247.2	5.352	9.156	0.9359	52.36
8	131.1	0.4882	0.2265	282.5	6.116	10.46	1.070	59.85
9	147.5	0.5492	0.2549	317.8	6.881	11.77	1.203	67.33

Conversion of Volumes or Capacities

	Liquid ounces to cu. cm.	Cu. cm. to liquid ounces	Pints to liters	Liters to pints	Quarts to liters	Liters to quarts	Gallons to liters	Liters to gallons	Bushels to hectoliters	Hectoliters to bushels
1	29.57	0.03381	0.4732	2.113	0.9464	1.057	3.785	0.2642	0.3524	2.838
2	59.15	0.06763	0.9464	4.227	1.893	2.113	7.571	0.5283	0.7048	5.676
3	88.72	0.1014	1.420	6.340	2.839	3.170	11.36	0.7925	1.057	8.513
4	118.3	0.1353	1.893	8.453	3.785	4.227	15.14	1.057	1.410	11.35
5	147.9	0.1691	2.366	10.57	4.732	5.283	18.93	1.321	1.762	14.19
6	177.4	0.2029	2.839	12.68	5.678	6.340	22.71	1.585	2.114	17.03
7	207.0	0.2367	3.312	14.79	6.625	7.397	26.50	1.849	2.467	19.86
8	236.6	0.2705	3.785	16.91	7.571	8.453	30.28	2.113	2.819	22.70
9	266.2	0.3043	4.259	19.02	8.517	9.510	34.07	2.378	3.172	25.54

Conversion of Masses

	Grains to grams	Grams to grains	Ounces (avoir.) to grams	Grams to ounces (avoir.)	Pounds (avoir.) to kilograms	Kilograms to pounds (avoir.)	Short tons (2000 lb.) to metric tons	Metric tons (1000 kg.) to short tons	Long tons (2240 lb.) to metric tons	Metric tons to long tons
1	0.06480	15.43	28.35	0.03527	0.4536	2.205	0.907	1.102	1.016	0.984
2	0.1296	30.86	56.70	0.07055	0.9072	4.409	1.814	2.205	2.032	1.968
3	0.1944	46.30	85.05	0.1058	1.361	6.614	2.722	3.307	3.048	2.953
4	0.2592	61.73	113.40	0.1411	1.814	8.818	3.629	4.409	4.064	3.937
5	0.3240	77.16	141.75	0.1764	2.268	11.02	4.536	5.512	5.080	4.921
6	0.3888	92.59	170.10	0.2116	2.722	13.23	5.443	6.614	6.096	5.905
7	0.4536	108.03	198.45	0.2469	3.175	15.43	6.350	7.716	7.112	6.889
8	0.5184	123.46	226.80	0.2822	3.629	17.64	7.257	8.818	8.128	7.874
9	0.5832	138.89	255.15	0.3175	4.082	19.84	8.165	9.921	9.144	8.857

Velocity Equivalents

(For conversion table see p. 80)

Centimeters per sec.	Meters per sec.	Meters per min.	Kilo-meters per hour	Feet per sec.	Feet per min.	Miles per hour	Knots
1	0.01	0.6 $\bar{1}$.77815	0.036 $\bar{2}$.55630	0.03281 $\bar{2}$.51598	1.9685 0.29414	0.02237 $\bar{2}$.34965	0.01942 $\bar{2}$.28825
100 2.00000	**1**	60 1.77815	3.6 0.55630	3.281 0.51598	196.85 2.29414	2.237 0.34965	1.942 0.28825
1.667 0.22184	0.01667 $\bar{2}$.22184	**1**	0.06 $\bar{2}$.77815	0.05468 $\bar{2}$.73783	3.281 0.51598	0.03728 $\bar{2}$.57150	0.03237 $\bar{2}$.51018
27.78 1.44370	0.2778 $\bar{1}$.44370	16.67 1.22184	**1**	0.9113 $\bar{1}$.95968	54.68 1.73783	0.6214 $\bar{1}$.79335	0.53960 $\bar{1}$.73207
30.48 1.48402	0.3048 $\bar{1}$.48402	18.29 1.26217	1.097 0.04032	**1**	60 1.77815	0.6818 $\bar{1}$.83367	0.59209 $\bar{1}$.77238
0.5080 $\bar{1}$.70586	0.005080 $\bar{3}$.70586	0.3048 $\bar{1}$.48402	0.01829 $\bar{2}$.26217	0.01667 $\bar{2}$.22185	**1**	0.01136 $\bar{2}$.05553	0.00987 $\bar{3}$.99423
44.70 1.65035	0.4470 $\bar{1}$.65035	26.82 1.42850	1.609 0.20670	1.467 0.16633	88 1.94448	**1**	0.86839 $\bar{1}$.93871
51.497 1.71178	0.51497 $\bar{1}$.71178	30.898 1.48993	1.8532 0.26793	1.68894 0.22761	101.337 2.00577	1.15155 0.06128	**1**

Mass Equivalents

(For conversion table see p. 77)

Kilograms	Grains	Ounces		Pounds		Tons		
		Troy and apoth.	Avoir-dupois	Troy and apoth.	Avoir-dupois	Short	Long	Metric
1	15432 4.18843	32.15 1.50719	35.27 1.54745	2.6792 0.42801	2.205 0.34333	0.0_21102 $\bar{3}$.04230	0.0_39842 $\bar{4}$.99309	0.001 $\bar{3}$.00000
0.0_46480 $\bar{5}$.81157	**1**	0.0_22083 $\bar{3}$.31876	0.0_22286 $\bar{3}$.35902	0.0_31736 $\bar{4}$.23958	0.0_31429 $\bar{4}$.15490	0.0_77143 $\bar{8}$.85387	0.0_76378 $\bar{8}$.80465	0.0_76480 $\bar{8}$.81157
0.03110 $\bar{2}$.49281	480 2.68124	**1**	1.09714 0.04026	0.08333 $\bar{2}$.92082	0.06857 $\bar{2}$.83614	0.0_43429 $\bar{5}$.53511	0.0_43061 $\bar{5}$.48590	0.0_43110 $\bar{5}$.49281
0.02835 $\bar{2}$.45255	437.5 2.64098	0.9115 $\bar{1}$.95974	**1**	0.07595 $\bar{2}$.88056	0.0625 $\bar{2}$.79588	0.0_43125 $\bar{5}$.49485	0.0_42790 $\bar{5}$.44563	0.0_42835 $\bar{5}$.45255
0.3732 $\bar{1}$.57199	5760 3.76042	12 1.07918	13.17 1.11944	**1**	0.8229 $\bar{1}$.91532	0.0_34114 $\bar{4}$.61429	0.0_33673 $\bar{4}$.56508	0.0_33732 $\bar{4}$.57199
0.4536 $\bar{1}$.65667	7000 3.84510	14.58 1.16386	16 1.20412	1.215 0.08468	**1**	0.0005 $\bar{4}$.69897	0.0_34464 $\bar{4}$.64975	0.0_34536 $\bar{4}$.65667
907.2 2.95770	140_6 7.14613	29167 4.46489	320_3 4.50515	2431 3.38571	2000 3.30103	**1**	0.8929 $\bar{1}$.95078	0.9072 $\bar{1}$.95770
1016 3.00691	15680_4 7.19535	326_3 4.51411	35840 4.55437	2722 3.43492	2240 3.35025	1.12 0.04922	**1**	1.016 0.00691
1000 3.00000	15432356 7.18843	32151 4.50719	35274 4.54745	2679 3.42801	2205 3.34333	1.102 0.04230	0.9842 $\bar{1}$.99309	**1**

The equivalents are given in the heavier type. Logarithms of the equivalents are given immediately below.

Subscripts after any figure, 0_3, 9_4, etc., mean that that figure is to be repeated the indicated number of times.

Pressure Equivalents

(For conversion table see p. 80)

Megabars or megadynes per sq. cm.	Kilograms per sq. cm. (Metric atmospheres)	Pounds per sq. in.	Short tons per sq. ft.	Atmospheres	Columns of mercury at temperature 0° C.		Columns of water at temperature 15° C.		
					Meters	Inches	Meters	Inches	Feet
1	**1.0197** 0.00848	**14.50** 1.16148	**1.044** 0.01882	**0.9869** $\bar{1}.99427$	**0.7500** $\bar{1}.87508$	**29.53** 1.47025	**10.21** 1.00886	**401.8** 2.60402	**33.48** 1.52484
0.9807 $\bar{1}.99152$	**1**	**14.22** 1.15300	**1.024** 0.01034	**0.9678** $\bar{1}.98579$	**0.7355** $\bar{1}.86660$	**28.96** 1.46177	**10.01** 1.00038	**394.0** 2.59555	**32.84** 1.51636
0.06895 $\bar{2}.83852$	**0.07031** $\bar{2}.84700$	**1**	**0.072** $\bar{2}.85733$	**0.06804** $\bar{2}.83279$	**0.05171** $\bar{2}.71360$	**2.036** 0.30876	**0.7037** $\bar{1}.84738$	**27.70** 1.44254	**2.309** 0.36336
0.9576 $\bar{1}.98119$	**0.9765** $\bar{1}.98966$	**13.89** 1.14267	**1**	**0.9450** $\bar{1}.97545$	**0.7182** $\bar{1}.85627$	**28.28** 1.45143	**9.773** 0.99004	**384.8** 2.58521	**32.06** 1.50603
1.0133 0.00573	**1.0333** 0.01421	**14.70** 1.16722	**1.058** 0.02955	**1**	**0.76** $\bar{1}.88081$	**29.92** 1.47598	**10.34** 1.01459	**407.2** 2.60976	**33.93** 1.53058
1.3333 0.12492	**1.3596** 0.13340	**19.34** 1.28640	**1.392** 0.14373	**1.316** 0.11919	**1**	**39.37** 1.59517	**13.61** 1.13378	**535.7** 2.72894	**44.64** 1.64976
0.03386 $\bar{2}.52975$	**0.03453** $\bar{2}.53823$	**0.4912** $\bar{1}.69124$	**0.03536** $\bar{2}.54857$	**0.03342** $\bar{2}.52402$	**0.02540** $\bar{2}.40484$	**1**	**0.3456** $\bar{1}.53861$	**13.61** 1.13378	**1.134** 0.05460
0.09798 $\bar{2}.99114$	**0.09991** $\bar{2}.99962$	**1.421** 0.15262	**0.1023** $\bar{1}.00996$	**0.09670** $\bar{2}.98541$	**0.07349** $\bar{2}.86622$	**2.893** 0.46139	**1**	**39.37** 1.59517	**3.281** 0.55198
0.002489 $\bar{3}.39598$	**0.002538** $\bar{3}.40446$	**0.03610** $\bar{2}.55746$	**0.002599** $\bar{3}.41479$	**0.002456** $\bar{3}.39024$	**0.001867** $\bar{3}.27106$	**0.07349** $\bar{2}.86622$	**0.02540** $\bar{2}.40484$	**1**	**0.08333** $\bar{2}.92082$
0.02986 $\bar{2}.47516$	**0.03045** $\bar{2}.48364$	**0.4332** $\bar{1}.63664$	**0.03119** $\bar{2}.49397$	**0.02947** $\bar{2}.46942$	**0.02240** $\bar{2}.35024$	**0.8819** $\bar{1}.94540$	**0.3048** $\bar{1}.48402$	**12** 1.07918	**1**

Energy or Work Equivalents

(For conversion table see p. 80)

Joules = 10^7 ergs	Kilogram-meters	Foot-pounds	Kilo-watt-hours	Cheval-vapeur-hours	Horse-power-hours	Liter-atmospheres	Kilo-gram-calories	British thermal units
1	**0.10197** $\bar{1}.00848$	**0.7376** $\bar{1}.86780$	**0.0_62778** $\bar{7}.44370$	**0.0_63777** $\bar{7}.57711$	**0.0_63725** $\bar{7}.57113$	**0.009869** $\bar{3}.99427$	**0.0_32390** $\bar{4}.37848$	**0.0_39486** $\bar{4}.97709$
9.80665 0.9915207	**1**	**7.233** 0.85932	**0.0_52724** $\bar{6}.43522$	**0.0_537037** $\bar{6}.56863$	**0.0_53653** $\bar{6}.56265$	**0.09678** $\bar{2}.98579$	**0.002344** $\bar{3}.37000$	**0.009302** $\bar{3}.96861$
1.356 0.13220	**0.1383** $\bar{1}.14068$	**1**	**0.0_63766** $\bar{7}.57590$	**0.0_651206** $\bar{7}.70932$	**0.0_650505** $\bar{7}.70333$	**0.01338** $\bar{2}.12647$	**0.0_33241** $\bar{4}.51068$	**0.001286** $\bar{3}.10929$
3.6×10^6 6.55630	**3.671×10^5** 5.56478	**2.655×10^6** 6.42410	**1**	**1.3596** 0.13342	**1.341** 0.12743	**35528** 4.55057	**860.5** 2.93478	**3415** 3.53339
2.648×10^6 6.42288	**270000.** 5.43136	**1.9529×10^6** 6.29068	**0.7355** $\bar{1}.86658$	**1**	**0.9863** $\bar{1}.99401$	**26131.** 4.41715	**632.9** 2.80135	**2512** 3.39996
2.6845×10^6 6.42887	**2.7375×10^5** 5.43735	**1.98×10^6** 6.29667	**0.7457** $\bar{1}.87257$	**1.0139** 0.00598	**1**	**26494** 4.42314	**641.7** 2.80735	**2547** 3.40595
101.33 2.00573	**10.333** 1.01421	**74.73** 1.87353	**0.0_42815** $\bar{5}.44943$	**0.0_43827** $\bar{5}.58284$	**0.0_43774** $\bar{5}.57686$	**1**	**0.02422** $\bar{2}.38425$	**0.09612** $\bar{2}.98281$
4183 3.62153	**426.6** 2.63000	**3086** 3.48932	**0.001162** $\bar{3}.06522$	**0.001580** $\bar{3}.19864$	**0.001558** $\bar{3}.19265$	**41.29** 1.61579	**1**	**3.968** 0.59861
1054 3.02291	**107.5** 2.03139	**777.52** 2.89071	**0.0_32928** $\bar{4}.46661$	**0.0_33981** $\bar{4}.60003$	**0.0_33927** $\bar{4}.59405$	**10.40** 1.01719	**0.25200** $\bar{1}.40139$	**1**

The equivalents are given in the heavier type. Logarithms of the equivalents are given immediately below.

Subscripts after any figure, 0_3, 9_4, etc., mean that that figure is to be repeated the indicated number of times.

1 cal = 4.186 Joules

Linear and Angular Velocity Conversion Factors

	Cm. per sec. to feet per min.	Feet per min. to cm. per sec.	Cm. per sec. to miles per hour	Miles per hour to cm. per sec.	Feet per sec. to miles per hour	Miles per hour to feet per sec.	Radians per sec. to rev. per min.	Rev. per min. to radians per sec.
1	1.97	0.508	0.0224	44.7	0.682	1.47	9.55	0.1047
2	3.94	1.016	0.0447	89.4	1.364	2.93	19.10	0.2094
3	5.91	1.524	0.0671	134.1	2.046	4.40	28.65	0.3142
4	7.87	2.032	0.0895	178.8	2.727	5.87	38.20	0.4189
5	9.84	2.540	0.1118	223.5	3.409	7.33	47.75	0.5236
6	11.81	3.048	0.1342	268.2	4.091	8.80	57.30	0.6283
7	13.78	3.556	0.1566	312.9	4.773	10.27	66.85	0.7330
8	15.75	4.064	0.1789	357.6	5.455	11.73	76.39	0.8378
9	17.72	4.572	0.2013	402.3	6.136	13.20	85.94	0.9425

Conversion of Pressures

	Pounds per sq. in. to kilograms per sq. cm.	Kilograms per sq. cm. to pounds per sq. in.	Atmospheres to pounds per sq. in.	Pounds per sq. in. to atmospheres	Atmospheres to kilograms per sq. cm.	Kilograms per sq. cm. to atmospheres
1	0.0703	14.22	14.70	0.0680	1.033	0.9678
2	0.1406	28.45	29.39	0.1361	2.067	1.936
3	0.2109	42.67	44.09	0.2041	3.100	2.903
4	0.2812	56.89	58.79	0.2722	4.133	3.871
5	0.3515	71.12	73.48	0.3402	5.166	4.839
6	0.4218	85.34	88.18	0.4082	6.200	5.807
7	0.4922	99.56	102.9	0.4763	7.233	6.774
8	0.5624	113.8	117.6	0.5443	8.266	7.742
9	0.6328	128.0	132.3	0.6124	9.300	8.710

Conversion of Energy, Work, Heat

	Ft.-lb. to kilo-gram-meters	Kilo-gram-meters to ft.-lb.	Ft.-lb. to B.t.u.	B.t.u. to ft.-lb.	Kilo-gram-meters to large calories	Large calories to kilo-gram-meters	Joules to small calories	Small calories to joules
1	0.1383	7.233	0.001286	777.5	0.002344	426.6	0.2390	4.183
2	0.2765	14.47	0.002572	1555.0	0.004688	853.2	0.4780	8.367
3	0.4148	21.70	0.003858	2333.0	0.007033	1280.0	0.7170	12.55
4	0.5530	28.93	0.005144	3110.0	0.009377	1706.0	0.9560	16.73
5	0.6913	36.16	0.006431	3888.0	0.01172	2133.0	1.195	20.92
6	0.8295	43.40	0.007717	4665.0	0.01407	2560.0	1.434	25.10
7	0.9678	50.63	0.009003	5443.0	0.01641	2986.0	1.673	29.28
8	1.106	57.86	0.01029	6220.0	0.01875	3413.0	1.912	33.47
9	1.244	65.10	0.01157	6998.0	0.02110	3839.0	2.151	37.65

Conversion of Power

	Horse powers to kilowatts	Kilowatts to horse powers	Metric horse powers to kilowatts	Kilowatts to metric horse powers	Horse powers to metric horse powers	Metric horse powers to horse powers
1	0.7457	1.341	0.7354	1.360	1.014	0.9863
2	1.491	2.682	1.471	2.719	2.028	1.973
3	2.237	4.023	2.206	4.079	3.042	2.959
4	2.983	5.364	2.942	5.439	4.056	3.945
5	3.728	6.705	3.677	6.799	5.069	4.932
6	4.474	8.046	4.413	8.158	6.083	5.918
7	5.220	9.387	5.148	9.518	7.097	6.904
8	5.965	10.73	5.884	10.88	8.111	7.890
9	6.710	12.07	6.619	12.24	9.125	8.877

Power Equivalents

(For conversion table see p. 80)

Horse power 550 standard ft.-lb. per sec.	Kilowatts (1000 joules per sec.)	Cheval-vapeur (metric h.p.)	Poncelets	M.-kg. per sec.	Ft.-lb. per sec.	Kg.-cal. per sec.	B.t.u per sec.
1	**0.7457** $\bar{1}$.87256	**1.014** 0.00599	**0.7604** $\bar{1}$.88105	**76.04** 1.88105	**550** 2.74036	**0.1783** $\bar{1}$.25104	**0.7074** $\bar{1}$.84965
1.341 0.12743	**1**	**1.360** 0.13343	**1.020** 0.00848	**102.0** 2.00848	**737.6** 2.86780	**0.2390** $\bar{1}$.37848	**0.9486** $\bar{1}$.97709
0.9863 $\bar{1}$.99402	**0.7355** $\bar{1}$.86659	**1**	**0.75** $\bar{1}$.87506	**75** 1.87506	**542.3** 2.73438	**0.1758** $\bar{1}$.24506	**0.6977** $\bar{1}$.84367
1.315 0.11896	**0.9807** $\bar{1}$.99152	**1.333** 0.12493	**1**	**100** 2.00000	**723.3** 2.85932	**0.2344** $\bar{1}$.37000	**0.9303** $\bar{1}$.96861
0.01315 $\bar{2}$.11896	**0.009807** $\bar{3}$.99152	**0.01333** $\bar{2}$.12493	**0.01** $\bar{2}$.00000	**1**	**7.233** 0.85932	**0.002344** $\bar{3}$.37000	**0.009303** $\bar{2}$.96861
0.00182 $\bar{3}$.25946	**0.001356** $\bar{3}$.13219	**0.00184** $\bar{3}$.26562	**0.00138** $\bar{3}$.14067	**0.1383** $\bar{3}$.14067	**1**	**0.0₃3241** $\bar{4}$.51068	**0.001286** $\bar{3}$.10929
5.610 0.74896	**4.183** 0.62153	**5.688** 0.75494	**4.266** 0.63000	**426.6** 2.63000	**3086** 3.48932	**1**	**3.968** 0.59861
1.414 0.15035	**1.054** 0.02291	**1.433** 0.15632	**1.075** 0.03139	**107.5** 2.03139	**777.5** 2.89071	**0.2520** $\bar{1}$.40138	**1**

The equivalents are given in the heavier type. Logarithms of the equivalents are given immediately below.

Subscripts after any figure, 0_3, 9_4, etc., mean that that figure is to be repeated the indicated number of times.

Density Equivalents and Conversion Factors

Equivalents					Conversion factors				
Grams per cu. cm.	Lb. per cu. in.	Lb. per cu. ft.	Short tons (2000 lb.) per cu. yd.	Lb. per U. S. gal.		Grams per cu. cm. to lb. per cu. ft.	Lb. per cu. ft. to grams per cu. cm.	Grams per cu. cm. to short tons per cu. yd.	Short tons per cu. yd. to grams per cu. cm.
1	**0.03613** $\bar{2}$.55787	**62.43** 1.79539	**0.8428** $\bar{1}$.92572	**8.345** 0.92143	1 2	62.43 124.90	0.01602 0.03204	0.8428 1.6860	1.186 2.373
27.68 1.44217	**1**	**1728** 3.23754	**23.33** 1.36792	**231** 2.36361	3 4	187.30 249.70	0.04806 0.06407	2.5280 3.3710	3.600 4.746
0.01602 $\bar{2}$.20466	**0.0₃5787** $\bar{4}$.76245	**1**	**0.0135** $\bar{2}$.13033	**0.1337** $\bar{1}$.12613	5 6	312.40 374.60	0.08009 0.09611	4.2140 5.0570	5.933 7.119
1.186 0.07428	**0.04286** $\bar{2}$.63205	**74.07** 1.86964	**1**	**9.902** 0.99572	7 8	437.00 499.40	0.11210 0.12820	5.9000 6.7420	8.306 9.492
0.1198 $\bar{1}$.07855	**0.004329** $\bar{3}$.63639	**7.481** 0.87396	**0.1010** $\bar{1}$.00432	**1**	9 10	561.90 624.30	0.14420 0.16020	7.5850 8.4280	10.680 11.870

Conversion of Heat Transmission and Conduction

	Small calories per sq. cm. to B.t.u. per sq. ft.	B.t.u. per sq. ft. to small calories per sq. cm.	Small calories per sq. cm. per cm. to B.t.u. per sq. ft. per in.	B.t.u. per sq. ft. per in. to small calories per sq. cm. per cm.	Small calories per sec. per sq. cm. per 1 deg. cent. per cm. thick, to B.t.u. per hr. per sq. ft. per 1 deg. fahr. per in. thick	B.t.u. per hr. per sq. ft. per 1 deg. fahr. per in. thick to small calories per sec. per sq. cm. per 1 deg. cent. per cm. thick
1	3.687	0.2712	1.451	0.6892	2.903×10^3	$0.0_{3}3445$
2	7.374	0.5424	2.902	1.378	5.806×10^3	$0.0_{3}6890$
3	11.06	0.8136	4.353	2.068	8.709×10^3	$0.0_{2}1034$
4	14.75	1.085	5.804	2.757	11.61×10^3	$0.0_{2}1378$
5	18.44	1.356	7.255	3.446	14.52×10^3	$0.0_{2}1722$
6	22.12	1.627	8.706	4.135	17.42×10^3	$0.0_{2}2067$
7	25.81	1.898	10.16	4.824	20.32×10^3	$0.0_{2}2412$
8	29.50	2.170	11.61	5.514	23.22×10^3	$0.0_{2}2756$
9	33.18	2.441	13.06	6.203	26.13×10^3	$0.0_{2}3100$

NOTE. 1 gram-calorie per sq. cm. = 3.687 B.t.u. per sq. ft.
1 gram-calorie per sq. cm. per cm. = 1.451 B.t.u. per sq. ft. per in.
1 gram-calorie per sec. per sq. cm. for a temp. grad. of 1 deg. cent. per cm.
= 360 kilogram-calories per hour per sq. m. for a temp. grad. of 1 deg. cent. per m.
= 2.903×10^3 B.t.u. per hour per sq. ft. for a temp. grad. of 1 deg. fahr. per in.

Values of Foreign Coins

(Legal standards: (*G*) = gold; (*S*) = silver)

Country	Monetary unit	Value in terms of U. S. money
Argentina (*G*)	Peso	$0.9647
Austria-Hungary (*G*)	Crown	0.2026
Belgium (*G* and *S*)	Franc	0.1929
Bolivia (*G*)	Boliviano	0.3893
Brazil (*G*)	Milreis	0.5463
British colonies in Australasia and Africa (*G*).	Pound sterling.	4.8665
Canada (*G*)	Dollar	1.0000
Central American States:		
Costa Rica (*G*)	Colon	0.4653
British Honduras (*G* or *S*).	Dollar	1.0000
Guatemala (*S*)	Peso	0.4446
Honduras (*S*)	Peso	0.4446
Salvador (*S*)	Peso	0.4446
Nicaragua (*S*)	Cordoba	1.0000
Chile (*G*)	Peso	0.3649
China (*S*)	Yuan	0.4777
Colombia (*G*)	Pound	4.8665
Denmark (*G*)	Crown	0.2680
Ecuador (*G*)	Sucre	0.4866
Egypt (*G*)	Pound	4.9429
Finland (*G*)	Markka	0.1929
France (*G* or *S*)	Franc	0.1929
German Empire (*G*)	Mark	0.2381

Country	Monetary unit	Value in terms of U. S. money
Great Britain (*G*)	Pound sterling	$4.8665
Greece (*G* and *S*)	Drachma	0.1929
Haiti (*G*)	Gourde	0.9647
India (British) (*G*)	Rupee	0.3244
Italy (*G* and *S*)	Lira	0.1929
Japan (*G*)	Yen	0.4984
Liberia (*G*)	Dollar	1.0000
Mexico (*G*)	Peso	0.4984
Netherlands (*G*)	Florin	0.4019
Norway (*G*)	Crown	0.2679
Panama (*G*)	Balboa	1.0000
Persia (*G* and *S*)	Kran	Variable
Peru (*G*)	Libra	4.8665
Philippine Islands (*G*)	Peso	0.5000
Portugal (*G*)	Escudo	1.0805
Roumania (*G*)	Leu	0.1929
Russia (*G*)	Ruble	0.5145
Santo Domingo (*G*)	Dollar	1.0000
Servia (*G*)	Dinar	0.1929
Siam (*G*)	Tical	0.3708
Spain (*G* and *S*)	Peseta	0.1929
Straits Settlement (*G*)	Dollar	0.5677
Sweden (*G*)	Crown	0.2679
Switzerland (*G*)	Franc	0.1929
Turkey (*G*)	Piaster	0.0439
Uruguay (*G*)	Peso	1.0340
Venezuela (*G*)	Bolivar	0.1929

TIME

Kinds of Time. Three kinds of time are recognized by astronomers, viz., sidereal, apparent solar, and mean solar time. The **sidereal day** is the interval between two consecutive transits of some fixed celestial object across any given meridian, or it is the interval required by the earth to make one complete revolution on its axis. This interval is constant but it is inconvenient as a time unit because the noon of the sidereal day occurs at all hours of the day and night. The **apparent solar day** is the interval between two consecutive transits of the sun across any given meridian. On account of the variable distance between the sun and earth, the variable speed of the earth in its orbit, the effect of the moon, etc., this interval is not constant and consequently cannot be kept by any simple mechanism, such as clocks or watches. To overcome the objection noted above, the **mean solar day** was devised. The mean solar day is the length of the average apparent solar day. Like the sidereal day it is constant, and like the apparent solar day its noon always occurs at approximately the same time of day. The astronomical day begins at mean solar noon and the hours run from one to twenty-four, while the civil day (mean solar) begins 12 hours earlier, at midnight, and the hours run from one to twelve, and then repeat from noon to midnight.

The Year. There are three different kinds of year used, the sidereal, the tropical, and the anomalistic. The **sidereal year** is the time taken by the earth to complete one revolution around the sun from a given star to the same star again. Its length is 365 days, 6 hours, 9 minutes, and 9 seconds. The **tropical year** is the time included between two successive passages of the vernal equinox by the sun, and since the equinox moves westward 50.″2 of arc a year, the tropical year is shorter by 20′23.″5 in time than the sidereal year. As the seasons depend upon the earth's position with respect to the equinox, the tropical year is the year of civil reckoning. The **anomalistic year** is the interval between two successive passages of the perihelion, namely, the time of the earth's nearest approach to the sun. The anomalistic year is only used in special calculations in astronomy.

The Calendar. The month depended originally upon the changes of the moon. The Mohammedan nations still use a lunar calendar with years of twelve lunar months, which alternately contain 355 and 356 days. According to their method of reckoning the same month falls in different seasons, and their calendars gain 1 year on ours every 33 years. The **Julian Calendar** (established 45 B. C.) discards all consideration of the moon and adopts 365¼ days as the true length of the year. It is still used in Russia and generally by the Greek Church. **Gregorian Calendar:** The true length of the tropical year is 365 days, 5 hr., 48 min., 45.5 sec., a difference of 11 min., 14.5 sec. by which the Julian year is too long. This amounts to a little more than 3 days in 400 years. To correct for this, those century years are made leap years which are divisible by 400 without remainder.

Standard Time. Prior to 1883 each city of the U. S. had its own time, which was determined by the time of passage of the sun across the local meridian. A system of standard time is used at present, according to which the United States, which extends from 65 deg. to 125 deg. West longitude, is divided into four sections, each of 15 deg. of longitude. The first or eastern section includes all territory between the Atlantic coast and an irregular line drawn from Detroit, Mich., through Pittsburg to Charleston, S. C., its most southern point. The time of this section is that of the 75-deg. meridian, which is 5

hr. slower than Greenwich time. The second (central) section includes all territory between the line mentioned, and an irregular line drawn from Bismarck, N. D., to the mouth of the Rio Grande. The third (mountain) section includes all territory between the last-named line and a line which passes through the western part of Idaho, Utah and Arizona. The fourth (Pacific) section covers the rest of the country to the Pacific Ocean. Standard time is uniform in each of these sections, but the time in one section differs by exactly 1 hr. from the section next to it. In cities situated on the border line of two sections, as, say, Pittsburg and Atlanta, the standard times of both sections are used, and in such cities when the time is given, it should be specified as eastern, central, etc. The system of standard time has been adopted in almost all civilized countries. All continental Europe, except Russia, uses a time 1 hr. faster than that of Greenwich; in Japan and Australia the time is 9 hr. faster.

TERRESTRIAL GRAVITY

By **standard gravity** is meant any locality where g_0 = 980.665 cm. per sec. per sec., or 32.1740 ft. per sec. per sec. This value, g_0, is assumed to be the value of g at sea level and latitude 45 deg.

Acceleration of Gravity

(U. S. Coast and Geodetic Survey, 1912)

Latitude, deg.	g		g/g_0	Latitude, deg.	g		g/g_0
	Cm./sec.²	Ft./sec.²			Cm./sec.²	Ft./sec.²	
0	978.0	32.088	0.9973	50	981.1	32.187	1.0004
10	978.2	32.093	0.9975	60	981.9	32.215	1.0013
20	978.6	32. 08	0.9979	70	982.6	32.238	1.0020
30	979.3	32.130	0.9986	80	983.1	32.253	1.0024
40	980.2	32.158	0.9995	90	983.2	32.258	1.0026

Correction for altitude above sea level: − 0.3 cm. per sec.² for each 1000 meters; − 0.003 ft. per sec.² for each 1000 feet.

SPECIFIC GRAVITY AND DENSITY

The **specific gravity of a solid or liquid** is the ratio of the mass of the body to the mass of an equal volume of water at some standard temperature. At the present time a temperature of 4 deg. cent. (39 deg. fahr.) is commonly used by physicists, but the engineer uses 60 deg. fahr. The specific gravity **of gases** is usually expressed in terms of hydrogen or air.

The **density** of a body is its mass per unit volume. If the gram is used as the unit of mass and the milliliter as the unit of volume, the figures representing the density are the same as the specific gravity of the body referred to water at 4 deg. cent. as unity. The customary unit is pounds per cu. ft.

The specific gravity of liquids is usually measured by means of an hydrometer (see p. 254).* Special arbitrary hydrometer scales are used in various trades and industries. The most common of these are the Baumé, Twaddell and Beck. Twaddell's hydrometer is used for liquids heavier than water. The number of degrees, N, which it indicates may be converted to specific gravities, G, by the formula $G = (5N + 1000)/1000$. The formula for the Beck hydrometer is $G = 170/(170 \pm N)$; for the Brix hydrometer $G = 400/(400 \pm N)$. In both of these the + sign is to be used for liquids lighter than water, the − sign for heavier liquids. For the salinometer (salometer), see p. 1734.* The specific gravities corresponding to the indications of the Baumé hydrometer are given in the following tables.

*See Marks' Mechanical Engineers' Handbook.

Specific Gravities at $\frac{60°}{60°}$ Fahr. Corresponding to Degrees Baumé for Liquids Lighter than Water

[Calculated from the formula, specific gravity $\frac{60°}{60°}$ fahr. $= \frac{140}{130 + \text{Deg. Bé.}}$]

Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity
10	1.0000	25	0.9032	40	0.8235	55	0.7568	70	0.7000	85	0.6512
11	0.9929	26	0.8974	41	0.8187	56	0.7527	71	0.6965	86	0.6482
12	0.9859	27	0.8917	42	0.8140	57	0.7487	72	0.6931	87	0.6452
13	0.9790	28	0.8861	43	0.8092	58	0.7447	73	0.6897	88	0.6422
14	0.9722	29	0.8805	44	0.8046	59	0.7407	74	0.6863	89	0.6393
15	0.9655	30	0.8750	45	0.8000	60	0.7368	75	0.6829	90	0.6364
16	0.9589	31	0.8696	46	0.7955	61	0.7330	76	0.6796	91	0.6335
17	0.9524	32	0.8642	47	0.7910	62	0.7292	77	0.6763	92	0.6306
18	0.9459	33	0.8589	48	0.7865	63	0.7254	78	0.6731	93	0.6278
19	0.9396	34	0.8537	49	0.7821	64	0.7216	79	0.6699	94	0.6250
20	0.9333	35	0.8485	50	0.7778	65	0.7179	80	0.6667	95	0.6222
21	0.9272	36	0.8434	51	0.7735	66	0.7143	81	0.6635	96	0.6195
22	0.9211	37	0.8383	52	0.7692	67	0.7107	82	0.6604	97	0.6167
23	0.9150	38	0.8333	53	0.7650	68	0.7071	83	0.6573	98	0.6140
24	0.9091	39	0.8284	54	0.7609	69	0.7035	84	0.6542	99	0.6114
										100	0.6087

Specific Gravities at $\frac{60°}{60°}$ Fahr. Corresponding to Degrees Baumé for Liquids Heavier than Water

[Calculated from the formula, specific gravity $\frac{60°}{60°}$ fahr. $= \frac{145}{145 - \text{Deg. Baumé}}$]

Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity	Degrees Baumé	Specific gravity
0	1.0000	12	1.0902	24	1.1983	36	1.3303	48	1.4948	60	1.7059
1	1.0069	13	1.0985	25	1.2083	37	1.3426	49	1.5104	61	1.7262
2	1.0140	14	1.1069	26	1.2185	38	1.3551	50	1.5263	62	1.7470
3	1.0211	15	1.1154	27	1.2288	39	1.3679	51	1.5426	63	1.7683
4	1.0284	16	1.1240	28	1.2393	40	1.3810	52	1.5591	64	1.7901
5	1.0357	17	1.1328	29	1.2500	41	1.3942	53	1.5761	65	1.8125
6	1.0432	18	1.1417	30	1.2609	42	1.4078	54	1.5934	66	1.8354
7	1.0507	19	1.1508	31	1.2719	43	1.4216	55	1.6111	67	1.8590
8	1.0584	20	1.1600	32	1.2832	44	1.4356	56	1.6292	68	1.8831
9	1.0662	21	1.1694	33	1.2946	45	1.4500	57	1.6477	69	1.9079
10	1.0741	22	1.1789	34	1.3063	46	1.4646	58	1.6667	70	1.9333
11	1.0821	23	1.1885	35	1.3182	47	1.4796	59	1.6860	......	....

Mohs's Scale of Hardness

1. Talc. 2. Gypsum. 3. Calc spar. 4. Fluor spar. 5. Apatite. 6. Feldspar. 7. Quartz. 8. Topaz. 9. Sapphire. 10. Diamond.

SECTION 2

MATHEMATICS

BY

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MATHEMATICS

BY

EDWARD V. HUNTINGTON

ARITHMETIC

NUMERICAL COMPUTATION

Number of Significant Figures. In any engineering computation, the data are ordinarily the results of measurement, and are correct only to a limited number of significant figures. Each of the numbers 3.840 and 0.003840 is said to be given "correct to four figures;" the true value lies in the first case between 3.8395 and 3.8405; in the second case, between 0.0038395 and 0.0038405. The **absolute error** is less than 0.001 in the first case, and less than 0.000001 in the second; but the **relative error** is the same in both cases, namely, an error of less than "one part in 3840."

If a number is written as 384000, the reader is left in doubt whether the number of correct significant figures is 3, 4, 5, or 6. This doubt can be removed by writing the number as 3.84×10^5 or 3.840×10^5 or 3.8400×10^5 or 3.84000×10^5.

In any numerical computation, the possible or desirable degree of accuracy should be decided on and the computation should then be so arranged that the required number of significant figures, and no more, is secured. Carrying out the work to a larger number of places than is justified by the data, is to be avoided, (1) because the form of the results leads to an erroneous impression of their accuracy, and (2) because time and labor are wasted in superfluous computation. The labor of working with six-place tables is nearly three times as great as that with four-place tables. In computations involving several steps, it is desirable to retain one extra figure until just before the final result is reached, in order to protect the last figure against the possible cumulative effect of small tabular errors. In **discarding superfluous figures,** if the first discarded figure is 5 or more, increase the preceding figure by 1. Thus, 3.14159, written correct to four figures, is 3.142; correct to three figures, 3.14. Again, 6.1297, correct to four figures, is 6.130.

Addition. In adding numbers, note that a doubtful final figure in any one number will render doubtful the whole column in which that figure lies; hence all figures to the right of that column are superfluous, and contribute nothing to the accuracy of the result.

```
  0.2056x
  2.572xx
 14.25xxx
576.1xxxx
---------
593.1
```

Subtraction. The "Austrian" or "shop" method is recommended. The mental process is as follows, the figures here printed in boldface type being the only ones written down:

[3 plus how many is 12?] 3 plus **9** is 12; 1 to carry.
[7 plus how many is 15?] 7 plus **8** is 15; 1 to carry.
5 plus **2** is 7. 8 plus **6** is 14.

```
14752
 8463
-----
 6289
```

This method is especially useful when it is desired to subtract from a given number the sum of several other numbers.

7 plus 1 is 8; plus 5 is 13; plus **9** is 22; 2 to carry.
5 plus 0 is 5; plus 2 is 7; plus **8** is 15; 1 to carry.
3 plus 1 is 4; plus 1 is 5; plus **2** is 7.
5 plus 3 is 8; plus **6** is 14.

$$\begin{array}{r} 14752 \\ \left.\begin{array}{r} 3125 \\ 101 \\ 5237 \end{array}\right] \\ \hline \mathbf{6289} \end{array}$$

The use of a wavy line to indicate subtraction is also recommended, as it will minimize the danger of adding when subtraction is intended.

Multiplication. In long examples in multiplication, the arrangement of work here illustrated is recommended, since it facilitates the abbreviation of the work by the omission, in practice, of all the figures on the right of the vertical line.

$$\begin{array}{r|l} 4956 & \\ 8372 & \\ \hline 39648 & \\ 1486 & 8 \\ 346 & 92 \\ 9 & 912 \\ \hline 41492 & \mathrm{xxx} \end{array}$$

The **position of the decimal point** should be determined by reference to the first, or left-hand, figures of the numbers, rather than by "pointing off" so-and-so many places from the right-hand end. For the right-hand figures of a number are the least important ones, and in many cases are entirely unknown (especially when the slide rule or a computing machine is used). The mental process for determining the decimal point is as follows:

(*a*) If the multiplier is a number like 3.1416, with only one figure preceding the decimal point, think of this number as "a little over 3;" then the product must be "a little over three times the number which is being multiplied;" and this gives the position of the decimal point at once, by inspection.

(*b*) If the multiplier is a number like 3141.6 [or 0.000 003 141 6], think of this number as "about 3, with the point moved three places to the right" [or "about 3, with the point moved six places to the left"]; then think what the answer would be if the multiplier were simply "about 3," and shift the decimal point accordingly.

Multiplication Tables. Crelle's large volume (Berlin, G. Reimer) gives the product of every three-figure number by every three-figure number; Peters's (Berlin, G. Reimer), of every four-figure number by every two-figure number. The smaller table of H. Zimmermann (Berlin, Wm. Ernst) gives the product of every three-figure number by every two-figure number.

Division. In long division, where the numbers are given only approximately, the work can be much abbreviated without loss of accuracy by "cutting off" one figure of the divisor at each step, instead of "bringing down" a doubtful zero in the dividend. Thus, 3.1416 ÷ 2.3026 = 1.3644.

$$\begin{array}{rl} 23026) & 31416(1 \\ & 23026 \\ \hline 2303) & 8390(3 \\ & 6909 \\ \hline 230) & 1481(6 \\ & 1380 \\ \hline 23) & 101(4 \\ & 92 \\ \hline 2) & 9(4 \end{array}$$

To determine the **position of the decimal point** in a problem of fractional division, shift the point (mentally) in both numerator and denominator (the same number of places in each) until the denominator is a number in the "standard form," that is, a number with only one figure preceding the decimal point. (This will not change the value of the fraction.) Then estimate the approximate magnitude of the quotient by inspection. Thus:

$$\frac{0.2718}{3141.6} = \frac{0.000\ 2718}{3.1416} = \text{“about 0.000 09”} = 0.000\ 08652;$$

$$\frac{31.416}{0.002718} = \frac{31\ 416.}{2.718} = \text{“about 10 000”} = 11\ 558.$$

Reciprocals. The reciprocal of N is $1/N$. Instead of dividing by a long number N, it is often better to multiply by the reciprocal of N. The table of reciprocals on pp. 24–27 gives the reciprocal of any number, correct to four figures. Barlow's Table (Spon & Chamberlain, New York) gives the reciprocal of every four-figure number correct to seven figures (but without facilities for interpolation). The reciprocals of numbers having more than four figures are best found by the use of a large table of logarithms.

Reciprocals of 1 ± x when x is Small.

$$1/(1+x) = 1 - x + [\text{error} < x^2, \text{ if } x \text{ is between 0 and 1}],$$
$$= 1 - x + x^2 - [\text{error} < x^3, \text{ if } x \text{ is between 0 and 1}].$$
$$1/(1-x) = 1 + x + [\text{error} < x^2 + 2x^3, \text{ if } x \text{ is between 0 and } \tfrac{1}{2}],$$
$$= 1 + x + x^2 + [\text{error} < x^3 + 2x^4, \text{ if } x \text{ is between 0 and } \tfrac{1}{2}].$$

NOTE. $1/(a \pm b) = (1/a)[1/(1 \pm x)]$, where $x = b/a$.

Notation by Powers of 10. All questions concerning the position of the decimal point are readily answered if each number is expressed in the "standard form," that is, as the product of two factors, one of which is a number with only one figure preceding the decimal point, while the other is a positive or negative power of 10. Thus, 3.1416×10^3 means 3.1416 with the point moved three places to the right, that is, 3141.6. Again, 3.1416×10^{-6} means 3.1416 with the point moved six places to the left, that is, 0.000 003 1416. This notation by powers of 10 should always be used in dealing with very large or very small numbers. Among electrical engineers its use is very general, even for numbers of moderate size.

Square Root. (*a*) If four figures of the root are sufficient, take the answer directly from the table of square roots, pp. 12–15. (*b*) To obtain a root of six or seven figures from the table, use the formula: $\sqrt{N} = a + [(N - a^2)/2a]$ (approx.), where a is the nearest value of $\sqrt{N}$ obtainable from the table, with three or four ciphers annexed. Here a^2 must be found exactly, by direct multiplication, so that at least three significant figures of the difference $N - a^2$ shall be known correctly; but this done, the division of $N - a^2$ by $2a$ should be carried to only three figures (logarithms or slide rule may be used).

NOTE. The simplest way to obtain any root of a seven-figure number correct to seven figures is to use a seven-place table of logarithms, if such a table is at hand.

Square Roots of 1 ± x when x is Small.

$$(1+x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}x - [\text{error less than } \tfrac{1}{8}x^2 \text{ if } 0 < x < 1]$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + [\text{error} < \tfrac{1}{16}x^3 \text{ if } 0 < x < 1]$$
$$(1-x)^{\frac{1}{2}} = 1 - \tfrac{1}{2}x - [\text{error} < \tfrac{1}{8}x^2 + \tfrac{1}{10}x^3 \text{ if } 0 < x < \tfrac{1}{2}]$$
$$= 1 - \tfrac{1}{2}x - \tfrac{1}{8}x^2 - [\text{error} < \tfrac{1}{16}x^3 + \tfrac{1}{16}x^4 \text{ if } 0 < x < \tfrac{1}{2}]$$

NOTE. $\sqrt{a+b} = \sqrt{a}\,(1+x)^{\frac{1}{2}}$, where $x = b/a$.

Cube Root. (*a*) If four figures of the root are sufficient, take the answer directly from the table of cube roots, pp. 16–21. (*b*) To obtain a root of six or seven figures from the table, use the formula: $\sqrt[3]{N} = a + [(N - a^3)/3a^2]$ (approx.), where a is the nearest value of $\sqrt[3]{N}$ obtainable from the table, with three or four ciphers annexed. Here a^3 must be found correct to seven or eight figures, by direct multiplication, so that at least three significant figures of the difference $N - a^3$ shall be known; but this done, the division of $N - a^3$ by $3a^2$ should be carried to only three or four figures (logarithms or the slide rule may be used).

NOTE. The simplest way to obtain any root of a seven-figure number correct to seven figures is to use a seven-place table of logarithms, if such a table is at hand.

Cube Roots of $1 \pm x$ when x is Small.

$$(1+x)^{1/3} = 1 + \tfrac{1}{3}x - [\text{error} < \tfrac{1}{9}x^2 \text{ if } 0 < x < 1],$$
$$= 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + [\text{error} < \tfrac{5}{81}x^3 \text{ if } 0 < x < 1],$$
$$(1-x)^{1/3} = 1 - \tfrac{1}{3}x - [\text{error} < \tfrac{1}{9}x^2 + \tfrac{1}{10}x^3 \text{ if } 0 < x < \tfrac{1}{2}],$$
$$= 1 - \tfrac{1}{3}x - \tfrac{1}{9}x^2 - [\text{error} < \tfrac{5}{81}x^3 + \tfrac{1}{15}x^4 \text{ if } 0 < x < \tfrac{1}{2}].$$

NOTE. $\sqrt[3]{a+b} = \sqrt[3]{a}(1+x)^{1/3}$, where $x = b/a$.

LOGARITHMS

Tables of Logarithms. The use of a table of logarithms greatly reduces the labor of multiplication, division, raising to powers, and extracting roots. The table on pp. 42–43 is carried out to four significant figures, and the following explanations should be sufficient to permit the use of the table readily, even by one without previous experience. For algebraic theory, see p. 113.

If more than four-figure accuracy is required, recourse must be had to a larger table. Five-place tables are available in great variety; the Macmillan Tables, 1913, are perhaps as convenient as any. If more than five figures are required, use Bremiker's six-place table, or proceed at once to a seven-place table: Schrön (Vieweg und Sohn, Braunschweig); Bruhns; Vega-Bremiker. If extreme accuracy is required, use the eight-place table by Bauschinger and Peters (Engelmann, Leipzig). Logarithmic paper, see p. 176.

To Find the Logarithm of Any Given (Positive) Number.

(*a*) WHEN THE GIVEN NUMBER IS BETWEEN 1 AND 10.

An inspection of the table on pp. 42–43 shows that as the number increases from 1 to 9.99. . . the logarithm of that number increases continuously from 0 to 0.999. . . For example, log 2.97 = 0.4728; log 2.98 = 0.4742.

If the given number contains four significant figures, it is necessary to interpolate between the tabulated values, as follows:

To find log 2.973, notice that this number is $\frac{3}{10}$ of the way from 2.97 to 2.98; hence its logarithm will be (approximately) $\frac{3}{10}$ of the way from 0.4728 to 0.4742. The difference here is 14 units, and $\frac{3}{10}$ of this difference is 4 (to the nearest unit); hence, by adding this 4 to 4728, log 2.973 = 0.4732. This process of interpolating should be performed mentally; the step of finding the tabular difference will be facilitated by a glance at the last column on the right, which gives, for each line of the table, the average of the differences along that line.

Again, to find log 4.098: From table, log 4.09 = 0.6117; adding $\frac{8}{10}$ of the difference (11), or about 9, gives: log 4.098 = 0.6126. Or better, since $\frac{8}{10}$ of the way forward is equal to $\frac{2}{10}$ of the way back, find in table log 4.10 = 0.6128, and *subtract* $\frac{2}{10}$ of 11, or 2, giving log. 4.098 = 0.6126. It should be noted that any interpolated value may be in error by 1 in the last place.

If the given number contains more than four significant figures, it should be cut down to four figures (see p. 88), since the later figures will not affect the result in four-place computations.

(*b*) WHEN THE GIVEN NUMBER IS LESS THAN 1 OR MORE THAN 10, it is simply necessary to notice that every such number can be regarded as obtainable from some number between 1 and 10 by merely shifting the decimal point (see p. 90); and that according to the rule at the foot of the table, moving the decimal point n places to the right [or left] in the number-column is equivalent to adding n [or $-n$] to the logarithm in the body of the table.

For example, to find log 2973. Here $2973 = 2.973 \times 10^3$ (*i.e.*, 2.973 with the decimal point moved 3 places to the right). From the table, log 2.973 = 0.4732. Hence, log 2973 = 0.4732 + 3, which may be written as 3.4732.

Again, to find log 0.0002973. Here $0.0002973 = 2.973 \times 10^{-4}$ (*i.e.*, 2.973 with the decimal point moved 4 places to the left). From the table, log 2.973 = 0.4732. Hence, log 0.0002973 = 0.4732 − 4. (This may be written as $\bar{4}.4732$, if desired, and is equal of course, to − 3.5268; this latter form, however, is not convenient in practice.)

It is thus evident that the logarithm of every positive number may be regarded as consisting of two parts: a decimal fraction, which is always positive (or zero); and a whole number, which may be positive, negative, or zero. The fractional part is called the **mantissa,** and is found from the table; the whole-number part is called the **characteristic,** and is determined by inspection.

To Find the Number Corresponding to a Given Logarithm.

(*a*) When the given logarithm is a positive decimal fraction (characteristic zero), simply reverse the process for finding the logarithm of a number between 1 and 10.

For example, given log N = 0.4732; to find N. In the body of the table it is seen that 0.4732 lies a little beyond 0.4728; hence N must lie a little beyond 2.97. By taking differences it is found that 4728 is in fact $4/14$ of the way from 0.4728 to the next higher logarithm; therefore N must be $4/14$ of the way from 2.97 to the next higher number. But $4/14$ of 1 is 0.3 (to the nearest tenth), hence N = 2.973.

Again, given log N = 0.6126; to find N. Here, 0.6126 is $9/11$ of the way from 0.6117 to the next higher logarithm; therefore N must be $9/11$ of the way from 4.09 to the next higher number. But $9/11$ of 1 is 0.8 (to the nearest tenth), hence N = 4.098.

(*b*) When the given logarithm has any given value (characteristic not zero), proceed as follows: First, be sure the given logarithm is in the "standard form," that is, a positive decimal fraction (mantissa) plus a positive or negative whole number (characteristic). For example, if log N is originally given in the form log N = − 3.5268, this must first be reduced to the (equivalent) form log N = 0.4732 − 4 (or $\bar{4}.4732$), before entering the table. Having the logarithm given in the standard form, suppose for the moment that the characteristic is zero, and find in the table the number corresponding to the given mantissa; then move the decimal point to the right or left according as the value of the characteristic is positive or negative.

For example, given log N = 0.4732 + 3; to find N. From the table, the number corresponding to 0.4732 is 2.973. The characteristic (+ 3) directs that the decimal point be moved 3 places to the right; hence $N = 2.973 \times 10^3 = 2973$.

Again, given log N = 0.4732 − 4; to find N. From the table, the number corresponding to 0.4732 is 2.973. The characteristic (− 4) indicates that the decimal point is to be moved 4 places to the left; hence $N = 2.973 \times 10^{-4} = 0.0002973$.

The number corresponding to a given logarithm is called its **antilogarithm.** Thus, if log 2973 = 0.4732 + 3, then 2973 = antilog (0.4732 + 3).

Note 1. In most tables of logarithms the decimal point is omitted, the tables being in fact not tables of logarithms, but tables of mantissas. This omission is of no consequence to the experienced computer, but is often perplexing to one who makes only occasional use of such tables.

Note 2. Many computers prefer to write negative characteristics in the form of some positive number minus some multiple of 10; thus, 0.4732 − 4 = 6.4732 − 10; 0.4732 − 13 = 7.4732 − 20; etc.

Fundamental Properties of Logarithms. The usefulness of logarithms in computation depends on the following properties:

$$(1)\ \log (ab) = \log a + \log b; \quad (3)\ \log (a^n) = n \log a;$$

$$(2)\ \log (a/b) = \log a - \log b; \quad (4)\ \log \sqrt[n]{a} = (1/n) \log a;$$

$$(5)\ \log 10^n = n$$

It is to be noted also that log 1 = 0, log 10 = 1, and log $(1/n) = -\log n$.

To Multiply by Logarithms. Find from the table the log. of each factor, and add; the result will be the log. of the product. Then find the product itself from the table.

EXAMPLE. To find $x = (4.098)(0.0002973)(72.1)$.

Answer: $x = 8.784 \times 10^{-2} = 0.08784$

$$
\begin{array}{lll}
\log\ 4.098 & = 0.6126 & \\
\log\ 0.0002973 & = 0.4732 - 4 & \\
\log\ 72.1 & = \underline{0.8579 + 1} & \\
\log\ x & = 1.9437 - 3 & = 0.9437 - 2.
\end{array}
$$

To Divide by Logarithms. First Method: Find from the table the log. of the numerator and the log. of the denominator, and subtract the second from the first; the result will be the logarithm of the quotient. Then find the quotient itself from the table.

EXAMPLE. To find $x = \dfrac{4.098}{0.0002973}$

Answer: $x = 1.378 \times 10^{4} = 13780$

$$
\begin{array}{ll}
\log\ 4.098 & = 0.6126 \\
\log\ 0.0002973 & = \underline{0.4732 - 4} \\
\log\ x & = 0.1394 + 4
\end{array}
$$

In order **to avoid negative mantissas** in cases where a larger mantissa would have to be subtracted from a smaller, modify the upper logarithm by adding and subtracting 1.

EXAMPLE. To find $x = \dfrac{0.0291}{63.4}$.

Answer: $x = 4.590 \times 10^{-4} = 0.0004590.$

$$
\begin{array}{lll}
\log 0.0291 & = 0.4639 - 2 & = 1.4639 - 3 \\
\log 63.4 & = \underline{0.8021 + 1} & = \underline{0.8021 + 1} \\
\log x & & = 0.6618 - 4
\end{array}
$$

But if the logarithms are written with the characteristics in front, and the "shop method" of subtraction is used (see p. 88), then no such special device is here required. Thus:

$$
\begin{array}{ll}
\log\ 0.0291 & = \bar{2}.4639 \\
\log\ 63.4 & = \underline{\bar{1}.8021} \\
\log\ x & = \bar{4}.6618
\end{array}
$$

To Divide by Logarithms. Second Method: Instead of subtracting the log. of a number, it is often convenient to add the **cologarithm** of that number; the colog. of N being defined by: $\operatorname{colog} N = \log (1/N) = -\log N$.

To find the colog. of a number, write the log. of the number in the standard form, and subtract it from $1.0000 - 1$, as in the following examples:

$$
\begin{array}{ll}
 & 1.0000 - 1 \\
\log 69.5 & = \underline{0.8420 + 1} \\
\operatorname{colog} 69.5 & = 0.1580 - 2
\end{array}
\qquad
\begin{array}{ll}
 & 1.0000 - 1 \\
\log 0.0002973 & = \underline{0.4732 - 4} \\
\operatorname{colog} 0.0002973 & = 0.5268 + 3
\end{array}
$$

This subtraction should be performed mentally. Thus, to subtract the mantissa, subtract each digit from 9 until the last non-zero digit is arrived at, and subtract this from 10; to subtract the characteristic, follow the regular rule of algebra ("reverse the sign and add"). Hence, if the logarithm itself is already written down, or can be read off from the table without interpolation, the cologarithm can be written down at once, by inspection. The use of cologarithms is not essential in logarithmic computation, but it often facilitates a compact arrangement of the work, especially in cases where the denominator of a fraction is itself the product of two or more factors.

To Find the nth Power of a Number by Logarithms. Find from the table the log. of the number, and multiply it by n; the result will be the logarithm of the nth power of that number. Then find the power itself from the tables.

EXAMPLE 1. Find $x = (0.0291)^3$

Answer: $x = 2.464 \times 10^{-5} = 0.00002464.$

$$
\begin{array}{lll}
\log 0.0291 & = 0.4639 - 2 & \\
 & \underline{\qquad\qquad 3} & \\
\log x & = 1.3917 - 6 & = 0.3917 - 5.
\end{array}
$$

EXAMPLE 2. Find $x = (0.0291)^{1.41}$
Answer: $x = 6.825 \times 10^{-3}$
$= 0.006825$

$$\begin{array}{lrr} \log 0.0291 = 0.4639 - 2 = & & -1.5361 \\ & & 1.41 \\ \hline & & 15361 \\ & & 61444 \\ & & 15361 \\ \hline \log x & = & -2.1659 \\ & = & 0.8341 - 3 \end{array}$$

To Find the nth Root of a Number by Logarithms. Find from the table the log. of the number, and divide it by n; the result will be the log. of the nth root of that number. Then find the root itself from the table.

EXAMPLE. Find $x = \sqrt[3]{4.098}$ $\qquad \log 4.098 = 0.6126$
Answer: $x = 1.600$ $\qquad \log x = 0.2042$

In order **to avoid fractional characteristics,** if the characteristic is not divisible by n, make it so divisible by adding and subtracting a suitable number before dividing.

EXAMPLE. Find $x = \sqrt[3]{0.0004590.}$
Answer: $x = 7.714 \times 10^{-2}$
$= 0.07714$

$$\begin{array}{r} \log 0.0004590 = 0.6618 - 4 \\ 3)\underline{2.6618 - 6} \\ \log x = 0.8873 - 2 \end{array}$$

But if the characteristic is positive, it is simpler to write it in front of the mantissa, and then divide directly.

THE SLIDE RULE

The slide rule is an indispensable aid in all problems in multiplication, division, proportion, squares, square roots, etc., in which a limited degree of accuracy is sufficient. The ordinary 10-in. Mannheim rule (see below) costs \$3 to \$4.50 and gives three significant figures correctly; the 20-in. rule (\$12.50) gives from three to four figures; the Fuller spiral rule (\$30) or the Thacher cylindrical rule (\$35) gives from four to five figures. For many problems the slide rule gives results more rapidly than a table of logarithms; it requires, however, more care in placing the decimal point in the answer. In all work with the slide rule, the position of the decimal point should be determined by inspection (see p. 89), only the sequence of digits being obtained from the instrument itself. Rapidity in the use of the instrument depends mainly on the skill with which the eye can estimate the values of the various divisions on the scale; expertness in this respect comes only with practice. The following explanations should be sufficient to permit the use of the ordinary slide rule successfully without previous experience and without knowledge of logarithms.

Multiplication and Division with a (Theoretical) Complete Logarithmic Scale. Consider a *complete* logarithmic scale (D, Fig. 1), assumed to extend indefinitely in both directions, only the main section, from 1 to 10, however, being usually available. Note that the divisions within the several sections are indentical, except that the numeral attached to each division of any one section is ten times the numeral attached to the corresponding division in the preceding section. [The distances laid off from 1 are proportional to the logarithms of the corresponding numbers, the distance from 1 to 10 being taken as unity.] Consider also a duplicate scale, C, numbered from 1 to 10, and arranged to slide along the fixed scale D as in the figures. By means of such a scale D, and slide C, any two numbers between 1 and 10 (and hence any two numbers whatever, with proper attention to the decimal point) can be multiplied or divided, as in the following examples.

To MULTIPLY 4 BY 6. In Fig. 1, starting with point 1 of the fixed scale, run the eye along from 1 to 4; then set the 1 of the slide opposite this point 4, and *run the eye forward along the slide from* 1 *to* 6; the point thus reached on the fixed scale is 24, which is equal to 4 × 6. This process gives the distance from 1 to 4 *plus* the distance from 1 to 6, and is, in fact, a mechanical method of adding the logarithms of these numbers; hence the result is the product of the numbers. Conversely,

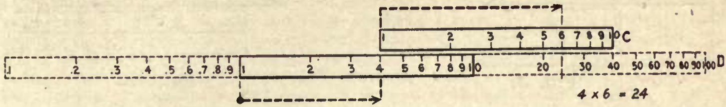


FIG. 1.

To DIVIDE 4 BY 6. In Fig. 2, starting with the point 1 of the fixed scale, run the eye along from 1 to 4; then set the 6 of the slide opposite the point 4, and *run the eye backward along the slide from* 6 *to* 1; the point thus reached on the fixed scale is 0.667, which is equal to 4 ÷ 6. This process gives the distance from 1 to 4 *minus* the distance from 1 to 6; and is, in fact, a mechanical method of subtracting the logarithms of these numbers; hence the result is their quotient.

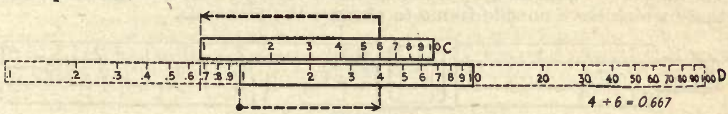


FIG. 2.

Multiplication and Division, Using Only a Single Section of the Scale. If only the main section of scale *D* is available (as is usually the case in practice), the result of multiplication may fall beyond the scale, as it does in Fig. 1. In such cases *divide the first factor by* 10 *before beginning to multiply;* this will bring the result within the scale, without affecting the sequence of digits.

For example, to multiply 4 by 6. Having found that the setting shown in Fig. 1 is not successful, *reset the slide as in Fig.* 3, *with* 10 *instead of* 1 *opposite* 4; run the eye backward along the slide from 10 to 1, thus reaching the (unrecorded) point corresponding to 4 ÷ 10; then, continuing from this point, run the eye forward along the slide from 1 to 6, as before; the point finally reached on the main scale is 2.4, which has the same sequence of digits as the required value 24. After a little practice, this preliminary step of dividing by 10 will be performed almost intuitively. Whether or not this step is necessary in any given case, can be determined only by trial.

The **general rule for multiplication** may be stated as follows, if preferred: To find the product of two factors, find one factor on the fixed scale; opposite this, set (tentatively) point 1 of the slide; on the slide find the second factor, and opposite this read the product on the main scale, if possible. If the product falls beyond the scale, begin over again, using point 10 of the slide instead of point 1.

In division also, the result may fall beyond the main section of the scale, as it does in Fig. 2. In such cases, it suffices merely to *multiply the result by* 10 in order to bring it within the scale; this will not affect the sequence of digits.

For example, to divide 4 by 6, set the slide as in Fig. 4, and follow out mentally the steps indicated by the arrows. It will be noticed that the supplementary step of multiplying by 10 is performed by simply running the eye along the slide from 1 to 10 without resetting the slide; for this reason, division on the slide rule is slightly easier than multiplication.

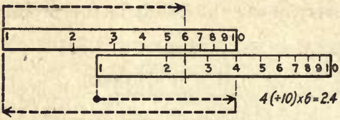


FIG. 3.

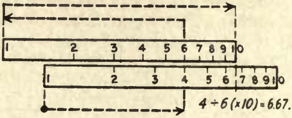


FIG. 4.

The Ordinary Mannheim Slide Rule has four scales, A, B, C, D, as shown in Fig. 5. Scales C and D are essentially the same as the C and D scales described above, and the principle just explained shows how they are used in multiplication and division. The fact that the D scale covers only the main section from 1 to 10 (all decimal points being omitted) is practically no restriction on the scope of the scale, as is seen in the preceding examples. A runner is provided, so that intermediate positions reached in the course of an extended computation may be indicated temporarily on the scale without the necessity of reading off their numerical values. The best runners are those which have no side frame to obscure the numerals.

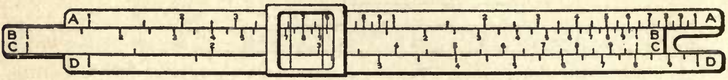


FIG. 5.

In problems involving successive multiplications and divisions, arrange the work so that multiplication and division are performed alternately.

For example, to calculate $\frac{a \times b \times c}{d \times e}$, divide the product $a \times b$ by d; multiply this quotient by c; and divide this product by e. Each operation will require only one shifting either of the slide (for multiplication) or of the runner (for division).

To multiply a number of different quantities by a *constant multiplier*, x, set the point 1 of slide opposite x, and read, by aid of the runner, the products of x by all the quantities which do not fall beyond the scale; then reset the slide, setting 10 instead of 1 opposite x, and read the products of x by all the remaining quantities.

To divide a number of different quantities by a *constant divisor*, y, first find (by the slide rule) the quotient $1 \div y$, and then use this as a constant multiplier.

Scales A and B are exactly like scales C and D, except that they cover two sections of the complete logarithmic scale, the graduations being only half as fine. Either pair of scales may be used for multiplication and division; C and D give more accurate readings, but have the disadvantage that in the case of multiplication the slide must often be shifted to the other end in order to keep the result on the scale—an inconvenience which is not present when the less accurate scales A and B are employed.

By the use of both pairs of scales, problems in squares and square roots may be readily solved; for every number on A, except for the decimal point, is the square of the number directly below it on D (use the runner).

A scale of sines, tangents, and logarithms is often printed on the back of the slide. For further details concerning the use of the slide rule in various problems, see the instruction books furnished with each instrument: Wm. Cox, "Manual of the Mannheim Slide Rule;" F. A. Halsey, "Manual of the Slide Rule;" etc.

Other Types of Slide Rules. The **duplex slide rule** ($5 to $18 according to length) shows on one face the regular A, B, C, D scales, and on the other face the scales A, B', C', D (where B' and C' are the same as B and C, only numbered in the reverse order), with a runner encircling the whole scale. This arrangement makes possible the solution of more complicated problems with fewer settings of the slide, but if the rule is to be used only for simple problems, the multiplicity of scales is rather confusing. Less complicated is the **polyphase rule**, which is like a Mannheim rule with the addition of a single inverted scale, C', printed in the middle of the slide. The **log log duplex slide rule** (10 in., $8) is especially adapted for handling complex problems involving fractional powers or roots, hyperbolic logarithms, etc. A number of **circular slide rules** are on the market, the best of which are operated by a milled thumbnut, like the stem wind of a watch. The advantage of the circular rule, aside from its compact size (some models are scarcely larger than a watch), lies in the fact that the scale is endless, so that the slide never has to be reset in order to bring the result within the scale. A disadvantage is found in the necessity of reading the figures in oblique positions, or else continually turning the instrument as a whole in the hand. The **Fuller and Thacher rules** already mentioned are invaluable for problems requiring greater accuracy than can be obtained with the ordinary rules. There are also many **special slide rules,** adapted to various special types of computation, such as calculating discharge of water through pipes, horse power of engines, dimensions of lumber, stadia measurements, etc. One of the most recent devices of this kind is the **Ross meridiograph** (L. Ross, San Francisco, Cal.), which is a circular slide rule for solving certain cases of spherical triangles. The **Eichhorn trigonometrical slide rule** solves any plane triangle.

COMPUTING MACHINES

For certain purposes computing machines have ceased to be luxuries and have become almost necessities; but they are expensive, and should be selected with reference to the special work which is to be done. The machines may be classified roughly into three groups, as follows:

Adding Machines, Non-listing. Of the machines of this kind, the most convenient in the hands of a careful operator is the well-known **Comptometer** (Felt & Tarrant Co., Chicago, Ill.; $250 to $350 according to size), or the recent **Burroughs non-listing** adding machine (Detroit, Mich., $175). To add a number, simply press a key in the proper column; the result appears on the dials in front of the keyboard. Multiplication as well as addition can be performed on this machine with great rapidity, and division also after a little practice. Weight, about 15 lb. Much less rapid, but less expensive and requiring somewhat less skill in operation, is the **Barrett adding machine** (Philadelphia, Pa.) with multiplying attachment. Other key-operated machines are the **Mechanical Accountant** (Providence, R. I.), and the **Austin** (Baltimore, Md.). The **American adding machine** (American Can Co., Chicago, Ill.; $39.50) is operated by pulling up a finger-lever for each digit. Small machines, operated by the use of a stylus, are the **Rapid computer** (Benton Harbor, Mich., $25); the **Gem** (Automatic Adding Machine Co., New York; $10), the **Arithstyle** (New York, $36) and the **Triumph** (Brooklyn, N. Y., $35). These machines, while much less rapid than the key-operated machines, are useful in simple addition. The **Underwood typewriter** is now supplied with a complete electrically driven adding machine attached, and the **Wahl adding attachment** is supplied on the Remington and other typewriters. **Ray Subtracto-Adder** (Richmond, Va., $25).

Adding and Listing Machines. The machines of this group not only add, but also print the items, totals and sub-totals. The **Burroughs** (Detroit, Mich.), the **Wales** (Adder Machine Co., Wilkes-Barre, Pa.), the **Comptograph** (Chicago, Ill.) and the **White** (New Haven, Conn.), resemble each other in having an 81-key keyboard; the **Dalton** (Cincinnati, Ohio) and the **Commercial** (White Adding Machine Co., New Haven,

Conn.) have a 10-key and a 9-key keyboard respectively, admitting of operation by the touch method. On all these machines, in order to add a number, first depress the proper keys and then pull a handle (or, in the case of electrically driven machines, press a button) to record the item. Multiplication cannot be performed conveniently, except on the Dalton. Subtraction can be performed only by adding the complement, except on the Commercial and on one type of the Burroughs. The prices range from $125 to $600, according to size and style, new models being constantly devised for special commercial purposes. A new and more portable machine of the 81-key type is the **Barrett** adding and listing machine (Philadelphia, Pa., $250). A cheaper machine, with a 10-key keyboard, is the **Standard** (St. Louis, Mo.). The new **American** adding and listing machine (American Can Co., Chicago, Ill.), operated by pulling up a finger-lever for each digit, costs only $88. The **Ellis** (Newark, N. J.) is an elaborate adding and listing machine having a complete typewriter incorporated with it. The **Elliott-Fisher bookkeeping machine** (Harrisburg, Pa.) and the **Moon-Hopkins billing machine** (St. Louis, Mo.) are intended primarily for commercial use; the latter is a complicated electric machine ($750) which combines many of the features of an adding and listing machine with those of a calculating machine.

Calculating Machines (so-called). Machines of this third group are intended primarily for multiplication and division; the types which have a keyboard can be used effectively for addition and subtraction also. They are all non-listing. The earliest commercially successful types were the Thomas and the Brunsviga. In both these types the multiplicand is set up by moving pegs in slots, or (in the newest models) by depressing keys, and the multiplication is effected by turning a handle for each digit of the multiplier—twice for a digit 2, three times for a digit 3, etc.; the result then appears on the dials. In the Thomas type the handle always turns in the same direction, the change from multiplication to division being effected by a shift key. In the Brunsviga type the handle is turned forward for multiplication and backward for division. Among the best examples of the Thomas type now on the American market are the **Tim**, with a single row of dials, the **Unitas**, with a double row of dials (both sold by Oscar Müller Co., New York City; also with keyboard and electric drive), and the **Reuter** (Philadelphia, Pa.). Prices, $300 upward. Another machine of this type, with keyboard, is the **Record** (U. S. Adding Machine Co., New York City). The **Brunsviga** is represented by Carl H. Reuter, Philadelphia, Pa.; various models. Of somewhat similar type are the **Triumphator** (New York City; $250), and **Colt's calculator** (Culmer Engineering Co., New York City). A new machine, on the same principle, but with keyboard, is the **Monroe** (made in Orange, N. J.; $250). The **Millionaire** (W. A. Morschhauser, New York City; $400), is from the mechanical point of view, the only true multiplying machine on the market (except the Moon-Hopkins). After the multiplicand is set up on the pegs, the digits of the multiplier are indicated successively by moving a pointer, the handle being turned only once for each digit. Further, the movement of the carriage is automatic. The newest models have keyboard and electric drive. The **Ensign electric calculating machine** (Boston, Mass.; $400) is a new machine with an 81-key keyboard on which it adds like an adding machine, and a secondary 10-key keyboard by means of which it multiplies and divides quite as rapidly as any of the calculating machines, the proper key being pressed just once for each digit of the multiplier. The **National calculator** (New York), and the **Lamb calculator** (Calculator Mfg. Co., New York) are less expensive machines devised for figuring payrolls and labor costs. A still simpler device for the same purpose is the **Calculacard** (New York). The machine called the **Calculagraph** (New York) is a time clock which automatically computes labor costs.

For graphical methods of computation, see pp. 106, 119, 170, 173–185.

FINANCIAL ARITHMETIC

For the facts which are commonly required in regard to compound interest, sinking funds, etc., see the headings of the tables on pp. 64–68.

ELEMENTARY GEOMETRY AND MENSURATION

GEOMETRICAL THEOREMS

(For geometrical constructions, see p. 101)

Right Triangles. $a^2 + b^2 = c^2$. (See Fig. 1). $\angle A + \angle B = 90°$. $p^2 = mn$. $a^2 = mc$. $b^2 = nc$. See also p. 105 and p. 132.

Oblique Triangles. (See also pp. 105, 134.) Sum of angles = 180°. An exterior angle = sum of the two opposite interior angles. (Fig. 1.)

The medians, joining each vertex with the middle point of the opposite side, meet in the center of gravity G (Fig. 2), which trisects each median.

The altitudes meet in a point called the orthocenter, O.

The perpendiculars erected at the midpoints of the sides meet in a point C, the center of the circumscribed circle. [In any triangle G, O, and C lie in line, and G is two-thirds of the way from O to C.]

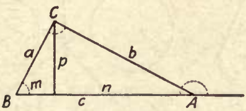


FIG. 1.

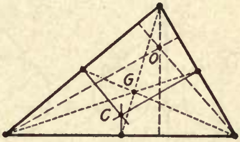


FIG. 2.

The bisectors of the angles meet in the center of the inscribed circle (Fig. 3).

The largest side of a triangle is opposite the largest angle; it is less than the sum of the other two sides, and greater than their difference.

FIG. 3.

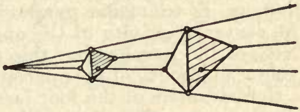


FIG. 4.

Similar Figures. Any two similar figures, in a plane or in space, can be placed in "perspective," that is, so that straight lines joining corresponding points of the two figures will pass through a common point (Fig. 4). That is, of two similar figures, one is merely an enlargement of the other. Assume that each length in one figure is k times the corresponding length in the other; then each area in the first figure is k^2 times the corresponding area in the second, and each volume in the first figure is k^3 times the corresponding volume in the second. If two lines are cut by a set of parallel lines (or parallel planes), the corresponding segments are proportional.

The Circle. (See also pp. 106, 137.) An angle inscribed in a semicircle is a right angle (Fig. 5). An angle inscribed in a circle, or an angle between a chord and a tangent, is measured by half the intercepted arc (Fig. 6). An angle formed by any two lines which meet a circle is measured by half the sum or half the difference of the intercepted arcs, according as the point of intersection of the lines lies inside (Fig. 7) or outside the circle (Fig. 8).

A tangent is perpendicular to the radius drawn to the point of contact.

If a variable line through A (Figs. 9 and 10) cuts a circle in P and Q, then

$\overline{AP} \times \overline{AQ}$ is constant; in particular, if A is an external point, $\overline{AP} \times \overline{AQ} = \overline{AT}^2$, where $\overline{AT}$ is the tangent from A.

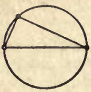


FIG. 5.

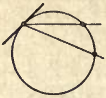


FIG. 6.


FIG. 7.

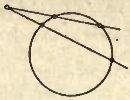


FIG. 8.

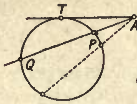


FIG. 9.

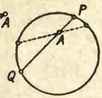


FIG. 10.

The radical axis (Fig. 11) of two circles is a straight line such that the tangents drawn from any point of this line to the two circles are of equal length. If the two circles intersect, the radical axis passes through their points of intersection. In any case, the radical axis bisects the common tangents of the two circles. The three radical axes of a set of three circles meet in a common point. (For equations, see p. 137.)

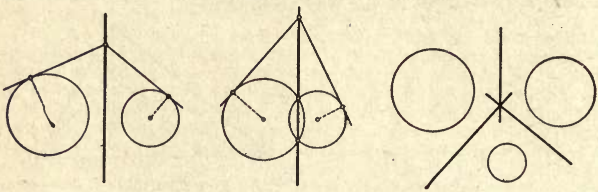


FIG. 11.

Dihedral Angles. The dihedral angle between two planes is measured by a plane angle formed by two lines, one in each plane, perpendicular to the edge (Fig. 12). (For solid angles, see p. 110.)

In a **tetrahedron,** or triangular pyramid, the four medians, joining each vertex with the center of gravity of the opposite face, meet in a point, the center of gravity of the tetrahedron; this point is ¾ of the way from any vertex to the center of gravity of the opposite face. The four perpendiculars erected at the circumcenters of the four faces meet in a point, the center of the circumscribed sphere. The four altitudes meet in a point called the orthocenter of the tetrahedron. The planes bisecting the six dihedral angles meet in a point, the center of the inscribed sphere.


FIG. 12.

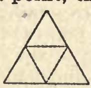


FIG. 13.

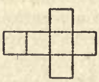


FIG. 14.

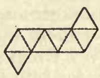


FIG. 15.

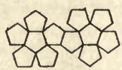


FIG. 16.

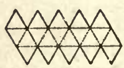


FIG. 17.

Regular Polyhedra (see also p. 110): Regular tetrahedron (Fig. 13), bounded by four equilateral triangles; cube (Fig. 14), bounded by six squares; octahedron (Fig. 15), bounded by eight equilateral triangles; dodecahedron (Fig. 16), bounded by twelve regular pentagons; icosahedron (Fig. 17), bounded by twenty equilateral triangles. Figs. 13–17 show how these solids can be made by cutting the surface out of paper and folding it together.

The Sphere. (See also p. 109.) If AB is a diameter, any plane perpendicular to AB cuts the sphere in a circle, of which A and B are called the poles. A great circle on the sphere is formed by a plane passing through the center. A spherical triangle is bounded by arcs of great circles (see p.

134). In two polar triangles, each angle in one is the supplement of the corresponding side in the other. In two symmetrical triangles, the sides and angles of one are equal to the corresponding sides and angles of the other, but arranged in the reverse order (like right-handed and left-handed gloves).

GEOMETRICAL CONSTRUCTIONS

To Bisect a Line AB (Fig. 18). (*a*) From *A* and *B* as centers, and with equal radii, describe arcs intersecting in *P* and *Q*, and draw *PQ*, which will bisect *AB* in *M*.

(*b*) Lay off $AC = BD$ = approximately half of *AB*, and then bisect *CD*.

To Draw a Parallel to a Given Line l Through a Given Point A (Fig. 19). With point *A* as center draw an arc just touching the line *l*; with any point *O* of the line as center, draw an arc *BC* with the same radius. Then a line through *A* touching this arc will be the required parallel. Or, use a straight edge and triangle. Or, use a sheet of celluloid with a set of lines parallel to one edge and about ¼ in. apart ruled upon it.

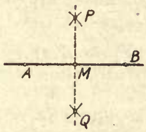


FIG. 18.

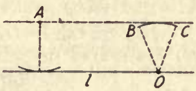


FIG. 19.

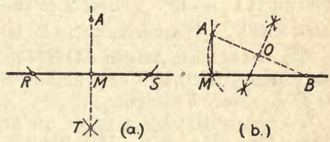


FIG. 20.

To Draw a Perpendicular to a Given Line from a Given Point A Outside the Line (Fig. 20). (*a*) With *A* as center, describe an arc cutting the line in *R* and *S*, and bisect *RS* in *M*. Then *M* is the foot of the perpendicular. (*b*) If *A* is nearly opposite one end of the line, take any point *B* of the line and bisect *AB* in *O*; then with *O* as center, and *OA* or *OB* as radius, draw an arc cutting the line in *M*. Or, (*c*) use a straight edge and triangle.

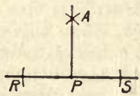


FIG. 21.

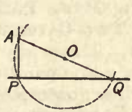


FIG. 22.

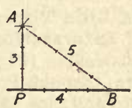


FIG. 23.

To Erect a Perpendicular to a Given Line at a Given Point P. (*a*) Lay off $PR = PS$ (Fig. 21), and with *R* and *S* as centers draw arcs intersecting at *A*. Then *PA* is the required perpendicular. (*b*) If *P* is near the end of the line, take any convenient point *O* (Fig. 22) above the line as center, and with radius *OP* draw an arc cutting the line in *Q*. Produce *QO* to meet the arc in *A*; then *PA* is the required perpendicular. (*c*) Lay off $PB = 4$ units of any scale (Fig. 23); from *P* and *B* as centers lay off $PA = 3$ and $BA = 5$; then *APB* is a right angle.

To Divide a Line AB into n Equal Parts (Fig. 24). Through *A* draw a line *AX* at any angle, and lay off *n* equal steps along this line. Connect the last of these divisions with *B*, and draw parallels through the other divi-

sions. These parallels will divide the given line into n equal parts. A similar method may be used to divide a line into parts which shall be proportional to any given numbers.

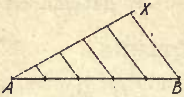


FIG. 24.

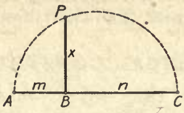


FIG. 25.

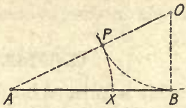


FIG. 26.

To Construct a Mean Proportional (or Geometric Mean) Between Two Lengths, m and n (Fig. 25). Lay off $AB = m$ and $BC = n$ and construct a semicircle on AC as diameter. Let the perpendicular erected at B meet the circumference at P. Then $BP = \sqrt{mn}$. (See p. 115.)

To Divide a Line AB in Extreme and Mean Ratio (the "golden section"). At one end, B, of the given line (Fig. 26), erect a perpendicular, BO, equal to half AB, and join OA. Along OA lay off $OP = OB$, and along AB lay off $AX = AP$. Then X is the required point of division; that is, $\overline{AX}^2 = AB \times BX$. Numerically, $AX = \frac{1}{2}(\sqrt{5} - 1)(AB) = 0.618(AB)$.

To Bisect an Angle AOB (Fig. 27). Lay off $OA = OB$. From A and B as centers, with any convenient radius, draw arcs meeting in M; then OM is the required bisector.

To draw the bisector of an angle when the vertex of the angle is not accessible (Fig. 28). Parallel to the given lines a, b, and equidistant from them, draw two lines a', b' which intersect; then bisect the angle between a' and b'.

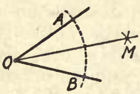


FIG. 27.

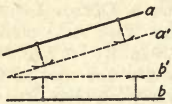


FIG. 28.

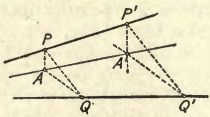


FIG. 29.

To Draw a Line Through a Given Point A and in the Direction of the Point of Intersection of Two Given Lines, when this point of intersection is inaccessible (Fig. 29). Draw any two parallel lines PQ and $P'Q'$ as in the figure; through P' draw a line parallel to PA, and through Q' draw a line parallel to QA; let these lines intersect in A', and draw the line AA'. This line AA' will (if produced) pass through the intersection of the two given lines.

To Construct, Approximately, the Length of a Circular Arc (Rankine). In Fig. 30 draw a tangent at A. Prolong the chord BA to C, making $AC = \frac{1}{2} AB$. With C as center, and radius CB, draw arc cutting the tangent in D. Then AD = arc AB, approximately (error about 4 min. in an arc of 60 deg.). Conversely, to find an arc AB on a given circle to equal a given length AD, take E one-fourth of the way from A to D, and with E as center and radius ED draw an arc cutting the circumference in B. Then arc $AB = AD$, approximately.

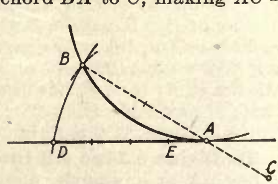


FIG. 30.

To Inscribe a Hexagon in a Circle (Fig. 31). Step around the circumference with a chord equal to the radius. Or, use a 60-deg. triangle.

To Circumscribe a Hexagon About a Circle (Fig. 32). Draw a chord AB equal to the radius. Bisect the arc AB in T. Draw the tangent at T (parallel to AB), meeting OA and OB in P and Q. Then draw a circle with radius OP or OQ and inscribe in it a hexagon, one side being PQ.

FIG. 31.

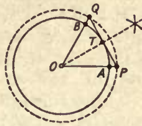


FIG. 32.

FIG. 33.

To Inscribe an Octagon in a Square (Fig. 33). From the corners as centers, and with radius equal to half the diagonal, draw four arcs, cutting the sides in eight points. The points will be the vertices of the octagon.

To Inscribe an Octagon in a Circle. Draw two perpendicular diameters, and bisect each of the quadrant arcs.

To Circumscribe an Octagon About a Circle. Draw a square about the circle, and draw the tangents to the circle at the points where the circle is cut by the diagonals of the square.

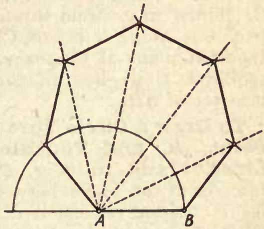


FIG. 34.

To Construct a Polygon of n Sides, One Side AB being Given (Fig. 34). With A as center and AB as radius, draw a semicircle, and divide it into n parts, of which $n - 2$ parts (counting from B) are to be used. Draw rays from A through these points of division, and complete the construction as in the figure (in which $n = 7$). Note that the center of the polygon must lie in the perpendicular bisector of each side.

To Draw a Tangent to a Circle from an external point A (Fig. 35). Bisect AC in M; with M as center and radius MC, draw arc cutting circle in P; then P is the required point of tangency.

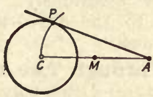


FIG. 35.

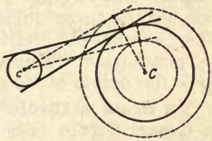


FIG. 36.

To Draw a Common Tangent to Two Given Circles (Fig. 36). Let C and c be the centers and R and r the radii ($R > r$). From C as center, draw two concentric circles with radii $R + r$ and $R - r$; draw tangents to these circles from c; then draw parallels to these lines at distance r. These parallels will be the required common tangents.

To Draw a Circle Through Three Given Points A, B, C, or to find the center of a given circular arc (Fig. 37). Draw the perpendicular bisectors of AB and BC; these will meet in the center, O.

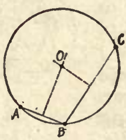


FIG. 37.

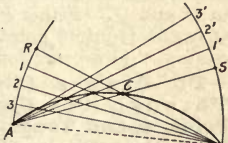


FIG. 38.

To Draw a Circular Arc Through Three Given Points When the Center is not Available (Fig. 38). With A and B as centers, and chord

AB as radius, draw arcs, cut by *BC* in *R* and by *AC* in *S*. Divide *RA* into *n* equal parts, 1, 2, 3, . . . Divide *BS* into the same number of equal parts, and continue these divisions at 1′, 2′, 3′, . . . Connect *A* with 1′, 2′, 3′, . . and *B* with 1, 2, 3, . . . Then the points of intersection of corresponding lines will be points of the required arc. (Construction valid only when $CA = CB$.)

To Draw a Circle Through Two Given Points, A, B, and Touching a Given Line, l (Fig. 39). Let *AB* meet line *l* in *C*. Draw any circle through *A* and *B*, and let *CT* be tangent to this circle from *C*. Along *l*, lay off *CP* and *CQ* equal to *CT*. Then either *P* or *Q* is the required point of tangency. (Two solutions.) Note that the center of the required circle lies in the perpendicular bisector of *AB*.

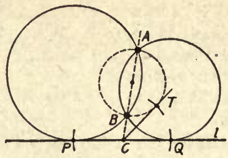


FIG. 39.

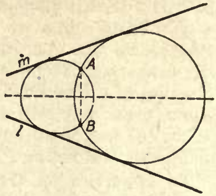


FIG. 40.

To Draw a Circle Through One Given Point, A, and Touching Two Given Lines, l and m (Fig. 40). Draw the bisector of the angle between *l* and *m*, and let *B* be the reflection of *A* in this line. Then draw a circle through *A* and *B* and touching *l* (or *m*), as in preceding construction. (Two solutions.)

To Draw a Circle Touching Three Given Lines (Fig. 41). Draw the bisectors of the three angles; these will meet in the center *O*. (Four solutions.) The perpendiculars from *O* to the three lines give the points of tangency.

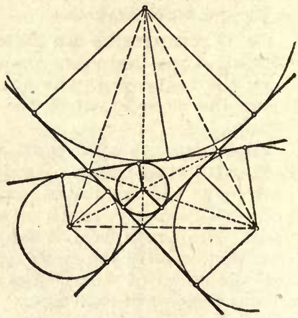


FIG. 41.

To Draw a Circle Through Two Given Points A, B, and Touching a Given Circle (Fig. 42). Draw any circle through *A* and *B*, cutting the given circle in *C* and *D*. Let *AB* and *CD* meet in *E*, and let *ET* be tangent from *E* to the circle just drawn. With *E* as center, and radius *ET*, draw an arc cutting the given circle in *P* and *Q*. Either *P* or *Q* is the required point of contact. (Two solutions.)

To Draw a Circle Through One Given Point, A, and Touching Two Given Circles (Fig. 43). Let *S* be a center of similitude for the two given circles, that is, the point of intersection of two external (or internal) common tangents. Through *S* draw any line cutting one circle in two points, the nearer of which shall be called *P*, and the other in two points, the more remote of which shall be called *Q*. Through *A*, *P*, *Q*

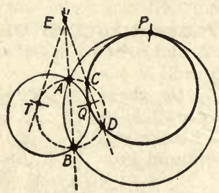


FIG. 42.

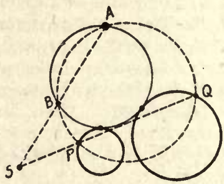


FIG. 43.

draw a circle cutting SA in B. Then draw a circle through A and B and touching one of the given circles (see preceding construction). This circle will touch the other given circle also. (Four solutions.)

To Draw an Annulus Which Shall Contain a Given Number of Equal Contiguous Circles (Fig. 44). (An annulus is a ring-shaped area enclosed between two concentric circles.) Let $R + r$ and $R - r$ be the inner and outer radii of the annulus, r being the radius of each of the n circles. Then the required relation between these quantities is given by $r = R \sin (180°/n)$, or $r = (R + r)[\sin (180°/n)]/[1 + \sin (180°/n)]$.

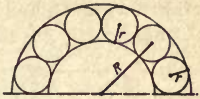


FIG 44.

For **methods of constructing ellipses and other curves,** see pp. 139–156.

LENGTHS AND AREAS OF PLANE FIGURES

Right Triangle (Fig. 45). $a^2 + b^2 = c^2$.
Area $= \frac{1}{2} ab = \frac{1}{2} a^2 \cot A = \frac{1}{2} b^2 \tan A = \frac{1}{4} c^2 \sin 2A$.

Equilateral Triangle (Fig. 46). Area $= \frac{1}{4} a^2 \sqrt{3} = 0.43301 a^2$.

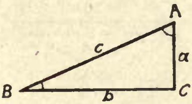


FIG. 45.

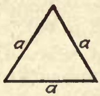


FIG. 46.

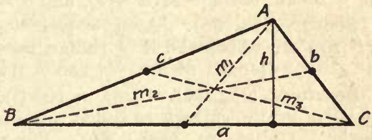


FIG. 47.

Any Triangle (Fig. 47). $s = \frac{1}{2}(a + b + c)$, $t = \frac{1}{2}(m_1 + m_2 + m_3)$,

$r = \sqrt{(s - a)(s - b)(s - c)/s}$ = radius inscribed circle,

$R = \frac{1}{2} a/\sin A = \frac{1}{2} b/\sin B = \frac{1}{2} c/\sin C$ = radius circumscribed circle;

Area $= \frac{1}{2}$ base $\times$ altitude $= \frac{1}{2} ah = \frac{1}{2} ab \sin C = rs = abc/4R$

$= \sqrt{s(s - a)(s - b)(s - c)} = \frac{4}{3} \sqrt{t(t - m_1)(t - m_2)(t - m_3)}$

$= r^2 \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C = 2R^2 \sin A \sin B \sin C$

$= \pm \frac{1}{2} \{ (x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) \}$, where

(x_1, y_1), (x_2, y_2), (x_3, y_3) are co-ordinates of vertices. See also p. 134.

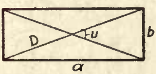


FIG. 48.

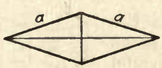


FIG. 49.

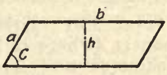


FIG. 50.

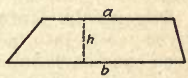


FIG. 51.

Rectangle (Fig. 48). Area $= ab = \frac{1}{2} D^2 \sin u$. [$u$ = angle between diagonals D, D.]

Rhombus (Fig. 49). Area $= a^2 \sin C = \frac{1}{2} D_1 D_2$. [$C$ = angle between two adjacent sides; D_1, D_2 = diagonals.]

Parallelogram (Fig. 50). Area $= bh = ab \sin C = \frac{1}{2} D_1 D_2 \sin u$. [$u$ = angle between diagonals D_1 and D_2; $D_1^2 + D_2^2 = 2(a^2 + b^2)$].

Trapezoid (Fig. 51). Area $= \frac{1}{2}(a + b)h = \frac{1}{2} D_1 D_2 \sin u$. [Bases a and b are parallel; u = angle between diagonals D_1 and D_2.]

Quadrilateral Inscribed in a Circle (Fig. 52). Area $= \frac{1}{2}D_1D_2 \sin u = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \frac{1}{2}(ac+bd)\sin u$; $s = \frac{1}{2}(a+b+c+d)$.

Any Quadrilateral (Fig. 53). Area $= \frac{1}{2}D_1D_2 \sin u$.

NOTE. $a^2+b^2+c^2+d^2 = D_1^2+D_2^2+4m^2$, where m = distance between midpoints of D_1 and D_2.

Polygons. See table, p. 39.

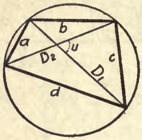


FIG. 52.

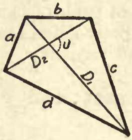


FIG. 53.

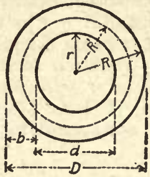


FIG. 54.

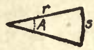


FIG. 55.

Circle. Area $= \pi r^2 = \frac{1}{2}Cr = \frac{1}{4}Cd = \frac{1}{4}\pi d^2 = 0.785398d^2$ (table, p. 30). Here r = radius, d = diam., C = circumference $= 2\pi r = \pi d$ (table, p. 28).

Annulus (Fig. 54). Area $= \pi(R^2 - r^2) = \pi(D^2 - d^2)/4 = 2\pi R'b$, where R' = mean radius $= \frac{1}{2}(R+r)$, and $b = R - r$.

Sector (Fig. 55). Area $= \frac{1}{2}rs = \pi r^2(A/360°) = \frac{1}{2}r^2$ rad A, where rad A = radian measure of angle A, and s = length of arc $= r$ rad A (table, p. 44).

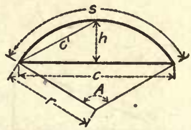


FIG. 56.

Segment (Fig. 56). Area $= \frac{1}{2}r^2$ (rad A − sin A) $= \frac{1}{2}[r(s-c) + ch]$, where rad A = radian measure of angle A (table, pp. 34–35, 44). For small arcs, $s = \frac{1}{3}(8c' - c)$, where c' = chord of half the arc. (Huygens's approximation.) NOTE. $c = 2\sqrt{h(d-h)}$; $c' = \sqrt{dh}$ or $d = c'^2/h$, where d = diameter of circle; $h = r(1 - \cos \frac{1}{2}A)$, $s = 2r$ rad $\frac{1}{2}A$.

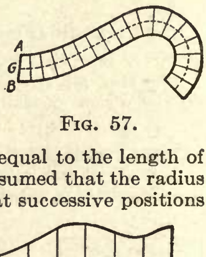


FIG. 57.

Ribbon bounded by two parallel curves (Fig. 57). If a straight line AB moves so that it is always perpendicular to the path traced by its middle point G, then the **area** of the ribbon or strip thus generated is equal to the length of AB times the length of the path traced by G. (It is assumed that the radius of curvature of G's path is never less than $\frac{1}{2}$ AB, so that successive positions of the generating line will not intersect.)

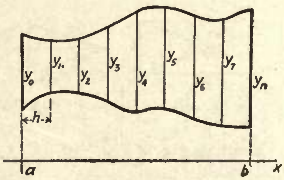


FIG. 58.

Simpson's Rule (Fig. 58). Divide the given area into n panels (where n is some even number) by means of $n+1$ parallel lines, called ordinates, drawn at constant distance h apart; and denote the lengths of these ordinates by $y_0, y_1, y_2, \ldots, y_n$. (Note that y_0 or y_n may be zero.) Then
Area $= \frac{1}{3}h[(y_0 + y_n) + 4(y_1 + y_3 + y_5 \ldots) + 2(y_2 + y_4 + y_6 \ldots)]$, approx. The greater the number of divisions, the more accurate the result. Note: Taking $y = f(x)$, where x varies from $x = a$ to $x = b$, and $h = (b-a)/n$, then the error $= -\frac{1}{180}\frac{(b-a)^5}{n^4}f''''(X)$, where $f''''(X)$ is the value of the fourth derivative of $f(x)$ for some (unknown) value, $x = X$, between a and b.

Ellipse (Fig. 59; see also p. 140). Area of ellipse $= \pi ab$. Area of shaded segment $= xy + ab \sin^{-1} (x/a)$. Length of perimeter of ellipse $= \pi(a + b)K$, where $K = [1 + \frac{1}{4}m^2 + \frac{1}{64}m^4 + \frac{1}{256}m^6 + \ . \ . \ .]$, $m = (a - b)/(a + b)$.

For $m =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$K =$	1.002	1.010	1.023	1.040	1.064	1.092	1.127	1.168	1.216	1.273

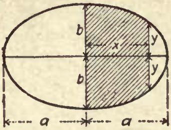


FIG. 59.

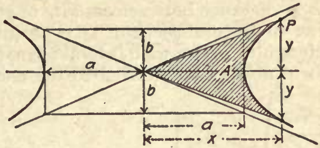


FIG. 60.

Hyperbola (Fig. 60; see also p. 144). In any hyperbola, shaded area $A = ab \log_e\left(\frac{x}{a} + \frac{y}{b}\right)$. In an equilateral hyperbola $(a = b)$, area $A = a^2 \sinh^{-1}(y/a) = a^2 \cosh^{-1}(x/a)$. For tables of hyperbolic functions, see p. 60. Here x and y are co-ordinates of point P.

Parabola (Fig. 61; see also p. 138). Shaded area $A = \frac{2}{3}ch$. In Fig. 62, length of arc $OP = s = \frac{1}{2}PT + \frac{1}{2}p \log_e \cot \frac{1}{2}u$. Here c = any chord; p = semi-latus rectum; PT = tangent at P. Note: $OT = OM = x$.

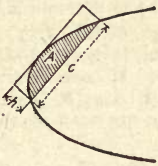


FIG. 61.

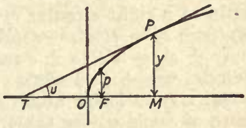


FIG. 62.

Other Curves. For lengths and areas, see pp. 147–156.

SURFACES AND VOLUMES OF SOLIDS

Regular Prism (Fig. 63). Volume $= \frac{1}{2}nrah = Bh$. Lateral area $= nah = Ph$. Here n = number of sides; B = area of base; P = perimeter of base.

Right Circular Cylinder (Fig. 64). Volume $= \pi r^2 h = Bh$. Lateral area $= 2\pi rh = Ph$. Here B = area of base; P = perimeter of base.

FIG. 63.

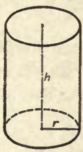


FIG. 64.

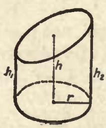


FIG. 65.

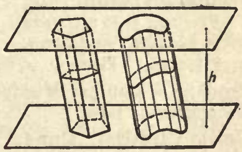


FIG. 66.

Truncated Right Circular Cylinder (Fig. 65). Volume $= \pi r^2 h = Bh$. Lateral area $= 2\pi rh = Ph$. Here h = mean height $= \frac{1}{2}(h_1 + h_2)$; B = area of base; P = perimeter of base.

Any Prism or Cylinder (Fig. 66). Volume $= Bh = Nl$. Lateral area $= Ql$. Here l = length of an element or lateral edge; B = area of base; N = area of normal section; Q = perimeter of normal section.

Any Truncated Prism or Cylinder (Fig. 67). Volume $= Nl$. Lateral area $= Qk$. Here l = distance between centers of gravity of areas of the two bases; k = distance between centers of gravity of perimeters of the two bases; N = area of normal section; Q = perimeter of normal section. For a truncated triangular prism with lateral edges a,b,c, $l = k = \frac{1}{3}(a + b + c)$. Note: l and k will always be parallel to the elements.

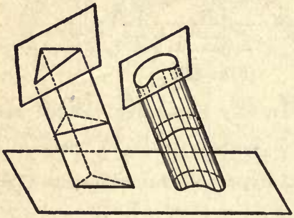


FIG. 67.

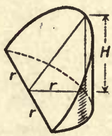


FIG. 68.

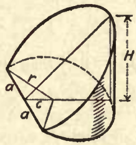


FIG. 69.

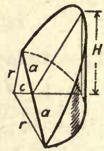


FIG. 70.

Special Ungula of a right circular cylinder. (Fig. 68.). Volume $= \frac{2}{3}r^2H$. Lateral area $= 2rH$. r = radius. (Upper surface is a semi-ellipse.)

Any Ungula of a right circular cylinder. (Figs. 69 and 70.) Volume $= H(\frac{2}{3}a^3 \pm cB)/(r \pm c) = H[a(r^2 - \frac{1}{3}a^2) \pm r^2 c \text{ rad } u]/(r \pm c)$. Lateral area $= H(2ra \pm cs)/(r \pm c) = 2rH(a \pm c \text{ rad } u)/(r \pm c)$. If base is greater (less) than a semicircle, use $+$ $(-)$ sign. r = radius of base; B = area of base; s = arc of base; u = half the angle subtended by arc s at center; rad u = radian measure of angle u (see table, p. 44).

Hollow Cylinder (right and circular). Volume $= \pi h(R^2 - r^2) = \pi hb(D - b) = \pi hb(d + b) = \pi hbD' = \pi hb\ (R + r)$. Here h = altitude; $r,R(d,D)$ = inner and outer radii (diameters); b = thickness $= R - r$; D' = mean diam. $= \frac{1}{2}(d + D) = D - b = d + b$.

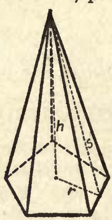


FIG. 71.

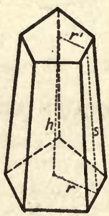


FIG. 72.

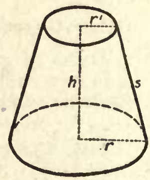


FIG. 73.

Regular Pyramid (Fig. 71). Volume $= \frac{1}{3}$ altitude $\times$ area of base $= \frac{1}{6}hran$. Lateral area $= \frac{1}{2}$ slant height $\times$ perimeter of base $= \frac{1}{2}san$. Here r = radius of inscribed circle; a = side (of regular polygon); n = number of sides; $s = \sqrt{r^2 + h^2}$. Vertex of pyramid directly above center of base.

Right Circular Cone. Volume $= \frac{1}{3}\pi r^2 h$. Lateral area $= \pi rs$. Here r = radius of base; h = altitude; s = slant height $= \sqrt{r^2 + h^2}$.

Frustum of Regular Pyramid (Fig. 72).
Volume $= \frac{1}{6}hran[1 + (a'/a) + (a'/a)^2]$.
Lateral area = slant height $\times$ half sum of perimeters of bases = slant height $\times$ perimeter of mid-section $= \frac{1}{2}sn(r + r')$. Here r,r' = radii

of inscribed circles; $s = \sqrt{(r - r')^2 + h^2}$; a, a' = sides of lower and upper bases; n = number of sides.

Frustum of Right Circular Cone (Fig. 73). Volume = $\frac{1}{3}\pi r^2 h[1 + (r'/r) + (r'/r)^2] = \frac{1}{3}\pi h(r^2 + rr' + r'^2) = \frac{1}{4}\pi h[(r + r')^2 + \frac{1}{3}(r - r')^2]$. Lateral area $= \pi s(r + r')$; $s = \sqrt{(r - r')^2 + h^2}$.

Any Pyramid or Cone. Volume $= \frac{1}{3}Bh$. B = area of base; h = perpendicular distance from vertex to plane in which base lies.

Any Pyramidal or Conical Frustum (Fig. 74). Volume = $\frac{1}{3}h(B + \sqrt{BB'} + B') = \frac{1}{3}hB[1 + (P'/P) + (P'/P)^2]$. Here B, B' = areas of lower and upper bases; P, P' = perimeters of lower and upper bases.

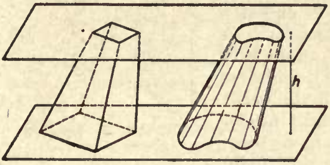


FIG. 74.

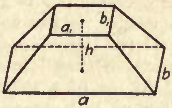


FIG. 75.

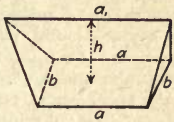


FIG. 76.

Obelisk (Frustum of a rectangular pyramid. Fig. 75).
Volume $= \frac{1}{6}h[(2a + a_1)b + (2a_1 + a)b_1] = \frac{1}{6}h[ab + (a + a_1)(b + b_1) + a_1 b_1]$.

Wedge (Rectangular base; a_1 parallel to a, a and at distance h above base. Fig. 76). Volume $= \frac{1}{6}hb(2a + a_1)$.

Sphere. Volume $= V = \frac{4}{3}\pi r^3 = 4.188790r^3 = \frac{1}{6}\pi d^3 = 0.523599d^3$ (table, p. 36) = $\frac{2}{3}$ volume of circumscribed cylinder. Area $= A = 4\pi r^2$ = four great circles (table, p. 30) $= \pi d^2 = 3.14159d^2$ = lateral area of circumscribed cylinder. Here r = radius; $d = 2r$ = diameter $= \sqrt[3]{6V/\pi} = 1.24070\sqrt[3]{V} = \sqrt{A/\pi} = 0.56419\sqrt{A}$.

Hollow Sphere, or spherical shell. Volume $= \frac{4}{3}\pi(R^3 - r^3) = \frac{1}{6}\pi(D^3 - d^3) = 4\pi R_1^2 t + \frac{1}{3}\pi t^3$. Here R, r = outer and inner radii; D, d = outer and inner diameters; t = thickness $= R - r$; R_1 = mean radius $= \frac{1}{2}(R + r)$.

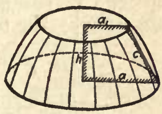


FIG. 77.

Spherical Segment of One Base. Zone (spherical "cap" of Fig. 78). Volume $= \frac{1}{6}\pi h(3a^2 + h^2) = \frac{1}{3}\pi h^2(3r - h)$ (table, p. 38). Lateral area (of zone) $= 2\pi rh = \pi(a^2 + h^2)$. Note: $a^2 = h(2r - h)$, where r = radius of sphere.

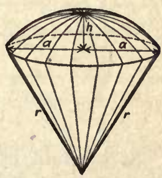


FIG. 78.

Any Spherical Segment. Zone (Fig. 77). Volume $= \frac{1}{6}\pi h(3a^2 + 3a_1^2 + h^2)$. Lateral area (zone) $= 2\pi rh$. Here r = radius of sphere. If the inscribed frustum of a cone be removed from the spherical segment, the volume remaining is $\frac{1}{6}\pi hc^2$, where c = slant height of frustum $= \sqrt{h^2 + (a - a_1)^2}$.

Spherical Sector (Fig. 78). Volume $= \frac{1}{3}r \times$ area of cap $= \frac{2}{3}\pi r^2 h$. Total area = area of cap + area of cone $= 2\pi rh + \pi ra$. Note: $a^2 = h(2r - h)$.

Spherical Wedge bounded by two plane semicircles and a **lune.** (Fig. 79.) Volume of wedge ÷ volume of sphere $= u/360°$. Area of lune ÷ area of sphere $= u/360°$. u = dihedral angle of the wedge.

Spherical Triangle bounded by arcs of three great circles. (Fig. 80.) Area of triangle $= \pi r^2 E/180° =$ area of octant $\times E/90°$. E = spherical excess $= 180° - (A + B + C)$, where A, B, and C are angles of the triangle. See also p. 134.

Solid Angles. Any portion of a spherical surface subtends what is called a **solid angle** at the center of the sphere. If the area of the given portion of spherical surface is equal to the square of the radius, the subtended solid angle is called a **steradian,** and this is commonly taken as the unit. The entire solid angle about the center is called a **steregon,** so that 4π steradians = 1 steregon. A so-called "solid right angle" is the solid angle subtended by a quadrantal (or trirectangular) spherical triangle, and a "spherical degree" (now little used) is a solid angle equal to $\frac{1}{90}$ of a solid right angle. Hence 720 spherical degrees = 1 steregon, or π steradians = 180 spherical degrees. If u = the angle which an element of a cone makes with its axis, then the solid angle of the cone contains $2\pi(1 - \cos u)$ steradians.

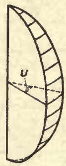


Fig. 79.

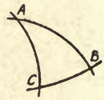


Fig. 80.

Regular Polyhedra. A = area of surface; V = volume; a = edge.

Name of solid (see p. 100)	Bounded by	A/a^2	V/a^3
Tetrahedron..................................	4 triangles	1.7321	0.1179
Cube..	6 squares	6.0000	1.0000
Octahedron..................................	8 triangles	3.4641	0.4714
Dodecahedron..............................	12 pentagons	20.6457	7.6631
Icosahedron.......	20 triangles	8.6603	2.1817

Ellipsoid (Fig. 81). Volume $= \frac{4}{3}\pi abc$, where a, b, c = semi-axes.

Spheroid (or ellipsoid of revolution). The volume of any segment made by two planes perpendicular to the axis of revolution may be found accurately by the prismoidal formula (p. 111).

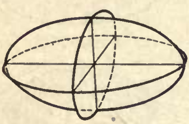


Fig. 81.

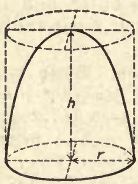


Fig. 82.

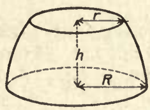


Fig. 83.

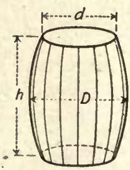


Fig. 84.

Paraboloid of Revolution (Fig. 82). Volume $= \frac{1}{2}\pi r^2 h = \frac{1}{2}$ volume of circumscribed cylinder.

Segment of Paraboloid of Revolution (Bases perpendicular to axis, Fig. 83). Volume of segment $= \frac{1}{2}\pi(R^2 + r^2)h$.

Barrels or Casks (Fig. 84). Volume $= \frac{1}{12}\pi h(2D^2 + d^2)$ approx. for circular staves. Volume $= \frac{1}{15}\pi h(2D^2 + Dd + \frac{3}{4}d^2)$ exactly for parabolic staves.

For a standing cask, partially full, compute contents by the prismoidal formula, p. 111. Roughly, the number of gallons, G, in a cask is given by $G = 0.0034n^2h$, where n = number of inches in the mean diameter, or $\frac{1}{2}(D + d)$, and h = number of inches in the height.

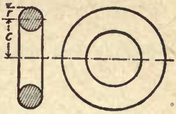


FIG. 85.

Torus, or Anchor Ring (Fig. 85). Volume = $2\pi^2cr^2$. Area = $4\pi^2cr$ (Proof by theorems of Pappus).

Theorems of Pappus. 1. Assume that a plane figure, area A, revolves about an axis in its plane but not cutting it; and let s = length of circular arc traced by its center of gravity. Then volume of the solid generated by A is $V = As$. For a complete revolution, $V = 2\pi rA$, where r = distance from axis to center of gravity of A.

2. Assume that a plane curve, length l, revolves about an axis in its plane but not cutting it; and let s = length of circular arc traced by its center of gravity. Then area of the surface generated by l is $S = ls$. For a complete revolution, $S = 2\pi rl$, where r = distance from axis to center of gravity of l.

NOTE. If V_1 or S_1 about any axis is known, then V_2 or S_2 about any parallel axis can be readily computed when the distance between the axes is known.

Generalized Theorems of Pappus. Consider any curved path of length s. If (1) a plane figure, area A [or (2) a plane curve, length l] moves so that its center of gravity slides along this curved path (Fig. 86), while the plane of A [or l] remains always perpendicular to the path, then (1) the volume generated by A is $V = As$ [and (2) the area generated by l is $S = ls$]. The path is assumed to curve so gradually that successive positions of A [or l] will not intersect.

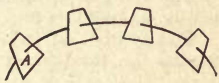


FIG. 86.

The Prismoidal Formula (Fig. 87). Volume $= \frac{1}{6}h(A + B + 4M)$, where h = altitude, A and B = areas of bases and M = area of a plane section midway between the bases. This formula is exactly true for any solid lying between two parallel planes and such that the area of a section at distance x from one of these planes is expressible as a polynomial of not higher than the third degree in x. It is approximately true for many other solids.

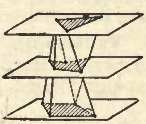


FIG. 87.

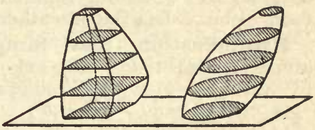


FIG. 88.

Simpson's Rule may be applied to finding volumes, if the ordinates y_1, y_2, be interpreted as the areas of plane sections, at constant distance h apart (p. 106).

Cavalieri's Theorem. Assume two solids to have their bases in the same plane. If the plane section of one solid at every distance x above the base is equal in area to the plane section of the other solid at the same distance x above the base, then the volumes of the two solids will be equal. See Fig. 88.

ALGEBRA

FORMAL ALGEBRA

Notation. The main points of separation in a simple algebraic expression are the $+$ and $-$ signs. Thus, $a + b \times c - d \div x + y$ is to be interpreted as $a + (b \times c) - (d \div x) + y$. In other words, the range of operation of the symbols $\times$ and $\div$ extends only so far as the next $+$ or $-$ sign. As between the signs $\times$ and $\div$ themselves, $a \div b \times c$ means, properly speaking, $a \div (b \times c)$; that is, the $\div$ sign is the stronger separative; but this rule is not always strictly followed, and in order to avoid ambiguity it is better to use the parentheses.

The range of influence of exponents and radical signs extends only over the next adjacent quantity. Thus, $2ax^3$ means $2a(x^3)$, and $\sqrt{2}ax$ means $(\sqrt{2})\ (ax)$. Instead of $\sqrt{2}ax$, it is safer, however, to write $\sqrt{2}\cdot ax$, or, better, $ax\sqrt{2}$.

Any expression within parentheses is to be treated as a single quantity. A horizontal bar serves the same purpose as parentheses.

The notation $a\cdot b$, or simply ab, means $a \times b$; and $a : b$, or a/b, means $a \div b$.

The symbol $|a|$ means the "absolute value of a," regardless of sign; thus, $|-2| = |+2| = 2$.

The symbol $n!$ (where n is a whole number) is read: "n factorial," and means the product of the natural numbers from 1 to n, inclusive. Thus $1! = 1$; $2! = 1 \times 2$; $3! = 1 \times 2 \times 3$; $4! = 1 \times 2 \times 3 \times 4$; etc.

The symbol $\neq$ or $\ne$ means "not equal to"; $\pm$ means "plus or minus."

The symbol $\approx$ is sometimes used for "approximately equal to."

Addition and Subtraction. $a + b = b + a$.

$(a + b) + c = a + (b + c)$. $a - (-b) = a + b$. $a - a = 0$.

$a + (x - y + z) = a + x - y + z$. $a - (x - y + z) = a - x + y - z$. A minus sign preceding a parenthesis operates to reverse the sign of every term within, when the parentheses are removed.

Multiplication and Simple Factoring. $ab = ba$. $(ab)c = a(bc)$. $a(b + c) = ab + ac$. $a(b - c) = ab - ac$. Also, $a \times (-b) = -ab$, and $(-a) \times (-b) = ab$; "unlike signs give minus; like signs give plus."

$(a + b)(a - b) = a^2 - b^2$.

$(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$.

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$; etc. (See table of binomial coefficients, p. 39; also p. 114.)

$a^2 - b^2 = (a - b)(a + b)$, $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1})$.

$a^n + b^n$ is factorable by $a + b$ only when n is odd; thus,

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$,

$a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$; etc.

The following transformation is sometimes useful:

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right].$$

Fractions. If m is not zero, $\dfrac{ma + mb + mc}{mx + my} = \dfrac{a + b + c}{x + y}$; that is, both numerator and denominator of a fraction may be multiplied or divided

by any quantity different from zero, without altering the value of the fraction.

To add two fractions, reduce each to a common denominator, and add the numerators: $\frac{a}{b} + \frac{x}{y} = \frac{ay}{by} + \frac{bx}{by} = \frac{ay + bx}{by}$.

To multiply two fractions: $\frac{a}{b} \times \frac{x}{y} = \frac{ax}{by}$; $\frac{a}{b} \times x = \frac{a}{b} \times \frac{x}{1} = \frac{ax}{b}$.

To divide one fraction by another, invert the divisor and multiply:

$$\frac{a}{b} \div \frac{x}{y} = \frac{a}{b} \times \frac{y}{x} = \frac{ay}{bx}; \quad \frac{a}{b} \div x = \frac{a}{b} \times \frac{1}{x} = \frac{a}{bx}.$$

Ratio and Proportion. The notation $a:b::c:d$, which is now passing out of use, is read: "a is to b as c is to d," and means simply $(a/b) = (c/d)$, or $ad = bc$. a and d are called the "extremes," b and c the "means," and d the "fourth proportional" to a, b, and c. The "mean proportional" between two numbers is the square root of their product; also called the "geometric mean" of the numbers (p. 115). If $a/b = c/d$, then $(a + b)/b = (c + d)/d$, and $(a - b)/b = (c - d)/d$; whence also, $(a + b)/(a - b) = (c + d)/(c - d)$. If $a/x = b/y = c/z = \ . \ . \ . = r$, then

$$(a + b + c + \ . \ . \ .)/(x + y + z + \ . \ . \ .) = r.$$

Variation. The notation $x \propto y$ is read: "x varies directly as y," or "x is directly proportional to y," and means $x = ky$, where k is some constant. To determine the constant k, it is sufficient to know any pair of values, as x_1 and y_1, which belong together; then $x_1 = ky_1$, and hence $x/x_1 = y/y_1$, or $x = (x_1/y_1)y$. The expression "x varies inversely as y," or "x is inversely proportional to y," means that x is proportional to $1/y$, or $x = k/y$.

Exponents. $a^{m+n} = a^m a^n$. $a^{m-n} = a^m/a^n$. $a^0 = 1$ (if $a \neq 0$). $a^{-m} = 1/a^m$. $(a^m)^n = a^{mn}$. $a^{1/n} = \sqrt[n]{a}$. Thus: $a^{1/2} = \sqrt{a}$, and $a^{1/3} = \sqrt[3]{a}$. $a^{m/n} = \sqrt[n]{a^m}$. Thus: $a^{2/3} = \sqrt[3]{a^2}$ and $a^{3/2} = \sqrt{a^3}$. $(\sqrt[n]{a})^n = a$. $(ab)^n = a^n b^n$. $(a/b)^n = a^n/b^n$. $(-a)^n = a^n$ if n is even. $(-a)^n = -a^n$ if n is odd. If n is positive and increases indefinitely, a^n becomes infinite if $a > 1$, and approaches 0 if $a < 1$ (a being always positive). Graphs, p. 174; series, p. 160.

Radicals. Except in the simple cases of square root and cube root, radical signs should always be replaced by fractional exponents: $\sqrt[n]{a} = a^{1/n}$. $(\sqrt[n]{a})^n = (a^{1/n})^n = a$. If n is odd, $\sqrt[n]{-a} = -\sqrt[n]{a}$; but if n is even, $\sqrt[n]{-a}$ is imaginary. Every positive number a has two square roots, one positive and the other negative; but the notation $\sqrt{a}$ always means the positive root; thus, $\sqrt{9} = 3$; $-\sqrt{9} = -3$. If the denominator of a fraction is of the form $\sqrt{a} \pm \sqrt{b}$, it is possible to "rationalize the denominator" by multiplying both numerator and denominator by $\sqrt{a} \mp \sqrt{b}$. Thus:

$$\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} - \sqrt{b}} = \frac{(\sqrt{a} + \sqrt{b})(\sqrt{a} + \sqrt{b})}{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})} = \frac{a + b + 2\sqrt{ab}}{a - b}$$

Logarithms. (For the use of logarithms in numerical computation, see p. 91.) The logarithm of a (positive) number N is the exponent of that power to which the base (10 or e) must be raised to produce N. Thus, $x = \log_{10} N$ means that $10^x = N$, and $x = \log_e N$ means that $e^x = N$. Logarithms to base 10 are called **common, denary,** or **Briggsian** logarithms. For table of 4-place common logarithms see pp. 40–43.

Logarithms to base e are called **hyperbolic, natural,** or **Napierian** logarithms. Here $e = 1 + 1 + 1/2! + 1/3! + 1/4! + \ldots = 2.718281828459\ldots$ For table of 4-place hyperbolic logarithms see pp. 58, 59.

If the subscript 10 or e is omitted, the base must be inferred from the context, the base 10 being used in numerical computation, and the base e in theoretical work. In either system,

$$\log(ab) = \log a + \log b \qquad \log(a^n) = n \log a \qquad \log 0 = -\infty$$
$$\log(a/b) = \log a - \log b \qquad \log(\sqrt[n]{a}) = (1/n)\log a \qquad \log 1 = 0$$
$$\log(1/n) = -\log n \qquad \log(\text{base}) = 1 \qquad \log\infty = \infty$$

The two systems are related as follows:

$\log_{10} e = M = 0.4342944819\ldots$; $\log_e 10 = 1/M = 2.3025850930\ldots$

$\log_{10} x = 0.4343 \log_e x$; $\log_e x = 2.3026 \log_{10} x$.

For tables of multiples of M and $1/M$, see p. 62. For graphs of the logarithmic and exponential functions, see p. 174; series, p. 160.

The Binomial Theorem. (For table of binomial coefficients, see p. 39 and p. 116.)

Let $(n)_1 = n$, $(n)_2 = \dfrac{n(n-1)}{1 \times 2}$, $(n)_3 = \dfrac{n(n-1)(n-2)}{1 \times 2 \times 3}$,

$$(n)_4 = \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}, \ldots$$

Then, for any value of n, provided $|x| < 1$,

$$(1 + x)^n = 1 + (n)_1 x + (n)_2 x^2 + (n)_3 x^3 + (n)_4 x^4 + \ldots$$

(If n is a positive integer, the series breaks off with the term in x^n, and is valid without restrictions on x, see p. 112.)

The most useful **special cases** are the following:

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots \quad (|x| < 1)$$

$$\sqrt[3]{1+x} = (1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \ldots \quad (|x| < 1)$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots \quad (|x| < 1)$$

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \frac{35}{128}x^4 - \ldots \quad (|x| < 1)$$

$$\frac{1}{\sqrt[3]{1+x}} = (1+x)^{-1/3} = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \frac{35}{243}x^4 - \ldots \quad (|x| < 1)$$

$$\sqrt{(1+x)^3} = (1+x)^{3/2} = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \frac{3}{128}x^4 - \ldots \quad (|x| < 1)$$

$$\frac{1}{\sqrt{(1+x)^3}} = (1+x)^{-3/2} = 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \frac{315}{128}x^4 - \ldots \quad (|x| < 1)$$

with corresponding formulæ for $\sqrt{1-x}$, etc., obtained by reversing the signs of the odd powers of x. Also, provided $|b| < |a|$:

$$(a+b)^n = a^n\left(1 + \frac{b}{a}\right)^n = a^n + (n)_1 a^{n-1}b + (n)_2 a^{n-2}b^2 + (n)_3 a^{n-3}b^3 + \ldots$$

where $(n)_1$, $(n)_2$, etc., have the values given above.

Arithmetical Progression. In an arithmetical progression, a; $a + d$; $a + 2d$; $a + 3d$; . . ., each term is obtained from the preceding term by adding a constant, called the constant difference, d. If n is the number of terms, the last term is $l = a + (n-1)d$; the "average" term is $\frac{1}{2}(a + l)$;

and the sum of the n terms is n times the average term, or $S = \frac{1}{2}n(a + l)$. The **arithmetical mean** between a and b is $(a + b)/2$.

Geometrical Progression. In a geometrical progression, a; ar; ar^2; ar^3; . . ., each term is obtained from the preceding term by multiplying by a constant, called the constant ratio, r. The nth term is ar^{n-1}. The sum of the first n terms is $S = a(r^n - 1)/(r - 1) = a(1 - r^n)/(1 - r)$. If r is a positive or negative fraction, that is, if $-1 < r < +1$, then r^n will approach zero as n increases, and the sum of n terms will approach $a/(1 - r)$ as a limit. The **geometric mean** between a and b is $\sqrt{ab}$; also called the **mean proportional** between a and b (p. 113; construction, p. 102).

The **harmonic mean** between a and b is $2ab/(a + b)$.

Summation of Certain Series by Second and Third Differences. Let a_1, a_2, a_3, . . . a_n be any series of n numbers, as in the first column of the adjoining scheme. By subtracting each number from the next following, form the column of "first differences," and by repeating this process, form the columns of second, third, etc., differences. If the kth differences are all equal, so that subsequent differences are all zero, the original series is called an arithmetical series of the kth order. In this special case the series can be summed as follows: Denote the numbers which stand at the head of the successive columns of differences by D', D'', D''', ... Then the nth term of the series is a_n, and the sum of the first n terms is S_n, where

Numbers	1st diff.	2nd diff.	3rd diff.
−64	37	−18	6
−27	19	−12	6
− 8	7	− 6	6
− 1	1	0	6
0	1	6	.
1	7	.	
8	.		
.			

$$a_n = a_1 + (n - 1)D' + \frac{(n-1)(n-2)}{1 \times 2}D'' + \frac{(n-1)(n-2)(n-3)}{1 \times 2 \times 3}D''' + \ldots$$

$$S_n = na_1 + \frac{n(n-1)}{1 \times 2}D' + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}D'' + \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}D''' + \ldots$$

If the series is, for example, of the third order, each of these formulæ will stop with the term involving D'''; and only a few terms of the series are required for the computation of the D's. (Differentials, p. 159.)

Sum of the Squares or Cubes of the First n Natural Numbers.

$$1 + 2 + 3 + \ldots + (n - 1) + n = \tfrac{1}{2}n(n + 1).$$
$$1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2 + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1).$$
$$1^3 + 2^3 + 3^3 + \ldots + (n - 1)^3 + n^3 = [\tfrac{1}{2}n(n + 1)]^2.$$

Formula for Interpolation by Second Differences. In any ordinary table giving a quantity y as a function of a variable x, let it be required to find the value of y corresponding to a value of x which is not given directly in the table, but which lies between two tabulated values, as x_1 and x_2. If $x = x_1 + md$, where $d = x_2 - x_1$ = the constant interval between two successive x's, and m is some proper fraction, then the corresponding value of y will be given by the formula

$$y = y_1 + mD' + \frac{m(m-1)}{1 \times 2}D'' + \frac{m(m-1)(m-2)}{1 \times 2 \times 3}D''' + \ldots$$

where D', D'', D''', . . . are the first, second, third, . . . differences in the

series of y's which begins with y_1 (see above), provided the function is of such a nature that the differences of higher orders become negligibly small.

The coefficients of D', D'', D''', . . . in the formula are the binomial coefficients for fractional values of m (see following table). The several terms of the formula (with careful attention to sign) are the successive corrections which must be added to y_1; the sum of these corrections should be rounded out to the nearest unit of the last significant place before adding. If $D' < 4$, the term involving D'', and later terms, can be neglected; the formula then reduces to $y = y_1 + mD'$, which is the familiar formula for ordinary, or "linear," interpolation. If $D''' < 8$ (or $D''\,'' < 12$, or $D''\,''' < 16$), the term involving D''' (or $D''\,''$, or $D''\,'''$) can be neglected.

Binomial Coefficients for Fractional Values of m

m	$(m)_2$	$(m)_3$	$(m)_4$	$(m)_5$
0.0	− 0.0000	0.0000	− 0.0000	0.0000
0.1	− 0.0450	0.0285	− 0.0207	0.0161
0.2	− 0.0800	0.0480	− 0.0336	0.0255
0.3	− 0.1050	0.0595	− 0.0402	0.0297
0.4	− 0.1200	0.0640	− 0.0416	0.0300
0.5	− 0.1250	0.0625	− 0.0391	0.0273
0.6	− 0.1200	0.0560	− 0.0336	0.0228
0.7	− 0.1050	0.0455	− 0.0262	0.0173
0.8	− 0.0800	0.0320	− 0.0176	0.0113
0.9	− 0.0450	0.0165	− 0.0087	0.0054

Here $(m)_2 = \frac{m(m-1)}{1 \times 2}$, $(m)_3 = \frac{m(m-1)(m-2)}{1 \times 2 \times 3}$, $(m)_4 = \frac{m(m-1)(m-2)(m-3)}{1 \times 2 \times 3 \times 4}$, etc. Compare p. 39.

Permutations. The number of possible permutations or arrangements of n different elements is $1 \times 2 \times 3 \times \ldots \times n = n!$ (read: "n factorial").

If among the n elements there are p equal ones of one sort, q equal ones of another sort, r equal ones of a third sort, etc., then the number of possible permutations is $(n!)/(p! \times q! \times r! \times \ldots)$, where $p + q + r + \ldots = n$.

Combinations. The number of possible combinations or groups of n elements taken r at a time (without repetition of any element within any one group), is $[n(n-1)(n-2)(n-3) \ldots (n-r+1)]/(r!) = (n)_r$. (See table of binomial coefficients, p. 39.) If repetitions are allowed, so that a group, for example, may contain as many as r equal elements, then the number of combinations of n elements taken r at a time is $(m)_r$, where $m = n + r - 1$. NOTE: $(n)_1 + (n)_2 + \ldots + (n)_n = 2^n - 1$.

SOLUTION OF EQUATIONS IN ONE UNKNOWN QUANTITY

Roots of an Equation. An equation containing a single variable x will in general be true for some values of x and false for other values. Any value of x for which the equation is true is called a **root** of the equation. To "solve" an equation means to find all its roots. Any root of an equation, when substituted therein for x, will "satisfy" the equation. An equation which is true for all values of x, like $(x + 1)^2 = x^2 + 2x + 1$, is called an **identity** [often written $(x + 1)^2 \equiv x^2 + 2x + 1$].

Types of Equations.

(*a*) Algebraic Equations:
of the first degree (linear), *e.g.*, $2x + 6 = 0$ (root: $x = -3$);
of the second degree (quadratic), *e.g.*, $x^2 - 2x - 3 = 0$ (roots: $-1, 3$);
of the third degree (cubic), *e.g.*, $x^3 - 6x^2 + 5x + 12 = 0$ (roots: $-1, 3, 4$).

(*b*) Transcendental Equations:
exponential equations, *e.g.*, $2^x = 32$ (root: $x = 5$); $2^x = -32$ (no root); trigonometric equations, *e.g.*, $10 \sin x - \sin 3x = 4$ (roots: 30°, 150°).

Legitimate Operations on Equations. An equation which is true for a particular value of x will remain true for that value of x after any one of the following operations is performed:

Adding any quantity to both sides; subtracting any quantity from both sides; transposing any term from one side to the other, provided its sign be changed; multiplying or dividing both sides by any quantity which is not zero; changing the signs of all the terms; raising both sides to any positive integral power; extracting any odd root of both sides; extracting any even root of both sides, provided the $\pm$ sign is used; taking the logarithms of both sides (both sides being positive); taking the sin, cos, tan, etc., of both sides.

Notice, however, that the new equation obtained by some of these operations may possess "additional roots" which did not belong to the original equation. This occurs especially when both sides are squared; thus, $x = -2$ has only one root, namely, -2; but $x^2 = 4$, obtained by squaring, has not only the root -2 but also another root, $+2$.

Equations of the First Degree (Linear Equations). Solution: Collect all the terms involving x on one side of the equation, thus: $ax = b$, where a and b are known numbers. Then divide through by the coefficient of x, obtaining $x = b/a$ as the root.

Equations of the Second Degree (Quadratic Equations). Solution: Throw the equation into the standard form $ax^2 + bx + c = 0$. Then the two roots are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The roots are real-and-distinct, coincident, or imaginary, according as $b^2 - 4ac$ is positive, zero, or negative. The sum of the roots is $x_1 + x_2 = -b/a$; the product of the roots is $x_1x_2 = c/a$.

Graphical Solution. Write the equation in the form $x^2 = px + q$, and plot the parabola $y_1 = x^2$, and the straight line $y_2 = px + q$. The abscissæ of the points of intersection will be the roots of the equation. If the line does not cut the parabola, the roots are imaginary.

Equations of the Third Degree with Term in x^2 Absent. Solution: After dividing through by the coefficient of x^3, any equation of this type can be written $x^3 = Ax + B$. Let $p = A/3$ and $q = B/2$. The general solution is as follows:

Case 1. $q^2 - p^3$ positive. One root is real, namely

$$x_1 = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}};$$

the other two roots are imaginary.

Case 2. $q^2 - p^3 =$ zero. Three roots real, but two of them equal.

$$x_1 = 2\sqrt[3]{q}, \; x_2 = -\sqrt[3]{q}, \; x_3 = -\sqrt[3]{q}.$$

Case 3. $q^2 - p^3$ negative. All three roots real and distinct. Determine an angle u between 0 and 180°, such that $\cos u = q/(p\sqrt{p})$. Then

$$x_1 = 2\sqrt{p}\cos(u/3),\; x_2 = 2\sqrt{p}\cos(u/3 + 120°),\; x_3 = 2\sqrt{p}\cos(u/3 + 240°).$$

Graphical Solution. Plot the curve $y_1 = x^3$, and the straight line $y_2 = Ax + B$. The abscissæ of the points of intersection will be the roots of the equation.

Equations of the Third Degree (General Case). Solution: The general cubic equation, after dividing through by the coefficient of the highest

power, may be written $x^3 + ax^2 + bx + c = 0$. To get rid of the term in x^2, let $x = x_1 - a/3$. The equation then becomes $x_1^3 = Ax_1 + B$, where $A = 3(a/3)^2 - b$, and $B = -2(a/3)^3 + b(a/3) - c$. Solve this equation for x_1, by the method above, and then find x itself from $x = x_1 - (a/3)$.

Graphical Solution. Without getting rid of the term in x^2, write the equation in the form $x^3 = -a[x + (b/2a)]^2 + [a(b/2a)^2 - c]$, and solve by the graphical method.

General Properties of Algebraic Equations. An algebraic equation of the nth degree in x is an equation of the type

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

where the a's are any given numbers (a_0 not zero), the expression on the left being called a **polynomial** of the nth degree in x. Such an equation will, in general, have n roots; but some of these n roots may be equal, and some may be imaginary. **Imaginary roots** always occur in pairs.

If the equation is written in the form: (a polynomial in x) = 0, then (1) if a is a root of the equation, $x - a$ is a factor of the polynomial; (2) if the polynomial can be factored in the form $(x - p)(x - q)(x - r) \ldots = 0$, each of the quantities $p, q, r, \ldots$ is a root of the equation; (3) if x is very large (either positive or negative), the higher powers of x are the most important; (4) if x is very small, the higher powers may be neglected.

Short Method of Substitution in a Polynomial. To find the value of $4x^4 - 14x^3 + 23x - 26$ when $x = 3$, for example, first arrange the terms in order of descending powers of x, and write the detached coefficients, with their signs, in a row, taking care to supply a zero coefficient for any missing term, including the constant term. Then, beginning at the left, bring down the first coefficient; multiply this by 3, and add to the second coefficient; multiply this result by 3 again, and add to the third coefficient; and so on.

4	− 14	0	23	− 26	(3
	12	− 6	− 18	15	
4	− 2	− 6	5	− 11	

The final result, − 11, is the value of the polynomial when $x = 3$.

Short Method of Dividing a Polynomial by x − a. The device just explained gives not only the value of the polynomial when $x = 3$, but also the result of dividing the polynomial by $x - 3$. Thus, in the case illustrated, the quotient is $4x^3 - 2x^2 - 6x + 5$ and the remainder is -11. That is, $4x^4 - 14x^3 + 0x^2 + 23x - 26 = (x - 3)(4x^3 - 2x^2 - 6x + 5) - 11$.

Exponential Equations. To solve an equation of the form $a^x = b$, take the logarithms of both sides: $x \log a = \log b$, whence $x = (\log b)/(\log a)$. For example, if $3^x = 0.4$, $x = \log 0.4/\log 3 = (0.6021 - 1)/0.4771 = -0.3979/0.4771 = -0.8340$. Notice that the complete logarithm must be taken, not merely the mantissa.

Trigonometric Equations. (1) To solve $a \cos x + b \sin x = c$, where a and b are positive: Find the acute angle u for which $\tan u = b/a$, and the angle v (between 0 and 180°) for which $\cos v = c/\sqrt{a^2 + b^2}$. Then $x_1 = u + v$ and $x_2 = u - v$ are roots of the equation.

(2) To solve $a \cos x - b \sin x = c$, where a and b are positive: Find u and v as above. Then $x_1 = -(u + v)$ and $x_2 = -(u - v)$ are roots of the equation.

General Method of Solution by Trial and Error. This method is applicable to a numerical equation of any form, and can be carried out to any desired degree of approximation. It is especially useful when a first approximation to a root is already known. Write the equation in the form

$f(x) = 0$, where $f(x)$ means any function of x, and plot the curve $y = f(x)$ for a sufficient number of values of x to obtain a general idea of the shape of the curve. Then pick out the regions in which the curve appears to cross the axis of x, and plot the curve more accurately in each of these regions. Thus, by successive approximations, plotting the important parts of the curve on a larger and larger scale, determine as accurately as necessary the points where the curve crosses the axis—that is, the values of x which make $f(x)$ equal to zero.

Thus, suppose that $f(x) = 3.0$ when $x = 2.6$ and -5.0 when $x = 2.7$ (see Fig. 1). Then the curve must cross the axis somewhere between $x = 2.6$ and $x = 2.7$; and since it will not vary greatly from a straight line between those points, it is seen that it must

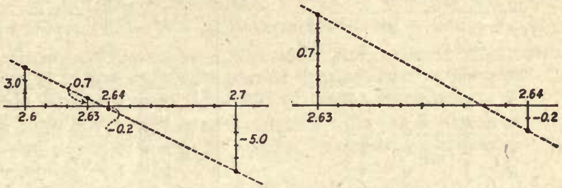


Fig. 1.

cross near 2.64. Suppose the value of $f(x)$ when computed for $x = 2.64$, is -0.2, and when computed for $x = 2.63$ is $+0.7$; then the root lies between $x = 2.63$ and 2.64. Plotting this section on the larger scale, it is seen that the next guess should be about 2.638; and so on.

Instead of writing the original equation with all the terms on the left-hand side, it is often better to divide the expression into two parts, say $f_1(x)$ and $f_2(x)$, writing the equation in the form $f_1(x) = f_2(x)$. If then the two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ be plotted separately, on the same diagram, the value of x corresponding to their point of intersection will be the desired root.

SOLUTION OF SIMULTANEOUS EQUATIONS

The Meaning of a System of Simultaneous Equations. To solve a system of n simultaneous equations in n unknowns, means to find all the sets of values of the unknowns (if any) which, when substituted in the given equations, will satisfy all the equations at the same time. If a system of equations has no solution, the equations are "inconsistent;" if it has an infinite number of solutions, the equations are "not all independent."

Simultaneous Equations of the First Degree in Two Unknowns.

	Factors	
(1) $a_1x + b_1y = c_1$	b_2	$-a_2$
(2) $a_2x + b_2y = c_2$	$-b_1$	a_1

$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2 \quad \therefore x = (b_2c_1 - b_1c_2)/(a_1b_2 - a_2b_1)$

$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1 \quad \therefore y = (a_1c_2 - a_2c_1)/(a_1b_2 - a_2b_1)$

Here (1) is multiplied by b_2, (2) by $-b_1$, and the products added so as to eliminate y; again, (1) is multiplied by $-a_2$, (2) by a_1, and the products added so as to eliminate x. (The process is most conveniently performed as follows: Write the multipliers, as b_2 and $-b_1$, at the right of the equations; multiply the first term of each equation by its proper multiplier and add; then multiply the second term of each equation by its proper multiplier, and add; and so on. This is simpler than the common practice of multiplying out each equation separately before adding.) If $a_1b_2 - a_2b_1 = 0$, the equations have no solution when $c_1 \neq c_2$, and an infinite number of solutions when

$c_1 = c_2$. The following **special solution** is possible when the sum and difference of the two unknowns are given:

$$\begin{array}{lll}
 & \text{Let } x + y = m & (1) \\
 & \text{and } x - y = n & (2) \\
(1) + (2): & 2x = m + n & \therefore x = \tfrac{1}{2}(m + n) \\
(1) - (2): & 2y = m - n & \therefore y = \tfrac{1}{2}(m - n)
\end{array}$$

Simultaneous Equations of the Second Degree in Two Unknowns. **(a)** When the product of the unknowns, and their sum or difference, are given:

$$\begin{array}{lrcl|r}
 & x + y & = & 5 & (1) \\
 & xy & = & 4 & (2) \\
\text{Squaring (1),} & x^2 + 2xy + y^2 & = & 25 & \\
\text{From (2),} & -4xy & = & -16 & \\
\text{Adding,} & x^2 - 2xy + y^2 & = & 9 & \\
\text{Hence,} & x - y & = & 3 \text{ or } -3 & \\
\text{But} & x + y & = & 5 \text{ or } \ \ 5 & \\
\text{Therefore} & x = 4,\ y = 1 & \text{or} & x = 1,\ y = 4 &
\end{array}
\qquad
\begin{array}{rcl|r}
x - y & = & 3 & (1) \\
xy & = & 4 & (2) \\
x^2 - 2xy + y^2 & = & 9 & \\
4xy & = & 16 & \\
x^2 + 2xy + y^2 & = & 25 & \\
x + y & = & 5 \text{ or } -5 & \\
x - y & = & 3 \text{ or } \ \ 3 & \\
x = 4,\ y = 1 & \text{or} & x = -1,\ y = -4 &
\end{array}$$

(b) When the product and the sum of the squares are given:

$$\begin{array}{lrcll}
 & xy & = & 5 & (1) \\
 & x^2 + y^2 & = & 26 & (2) \\
\text{From (1),} & 2xy & = & 10 & (3) \\
(2) + (3): & x^2 + 2xy + y^2 & = & 36 & (4) \\
(2) - (3): & x^2 - 2xy + y^2 & = & 16 & (5)
\end{array}
\qquad
\begin{array}{lcccc}
\sqrt{(4)}: x + y = & 6 \text{ or} & 6 \text{ or} & -6 \text{ or} & -6 \\
\sqrt{(5)}: x - y = & 4 \text{ or} & -4 \text{ or} & 4 \text{ or} & -4 \\
\hline
\therefore x = & 5 & 1 & -1 & -5 \\
\therefore y = & 1 & 5 & -5 & -1
\end{array}$$

(with "or" between each pair of solutions)

(c) When the sum or difference, and the sum of the squares, are given:

$$\begin{array}{lrcll}
 & x + y & = & 5 & (1) \\
 & x^2 + y^2 & = & 17 & (2) \\
(1)^2: & x^2 + 2xy + y^2 & = & 25 & \\
(2): & x^2 \qquad + y^2 & = & 17 & \\
(1)^2 - (2): & 2xy & = & 8 & \\
 & xy & = & 4 &
\end{array}
\qquad
\begin{array}{lrcll}
 & x - y & = & 3 & (1) \\
 & x^2 + y^2 & = & 17 & (2) \\
(1)^2: & x^2 - 2xy + y^2 & = & 9 & \\
(2): & x^2 \qquad + y^2 & = & 17 & \\
(1)^2 - (2): & -2xy & = & -8 & \\
 & xy & = & 4 &
\end{array}$$

Then proceed as in case (*a*), above. | Then proceed as in case (*a*), above.

(d) When one equation is of the first degree and the other of the second, as $ax + by = c$, and $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$: Solve the first equation for y in terms of x, and substitute in the second. This will give a quadratic equation in x. Solve this quadratic for the two values of x, and for each of these values of x find the corresponding value of y by substituting in the equation of the first degree.

Simultaneous Equations of the First Degree in n Unknowns. For example:

		Factors		
(*a*)	$2x - y + 3z + 5w = 29$	3	1	2
(*b*)	$5x + 2y - 2z + 3w = 15$	−5		
(*c*)	$3x - 4y + 7z - w = 12$		5	
(*d*)	$4x + 3y - 5z + 2w = 3$			−5
(*e*)	$-19x - 13y + 19z = 12$	−2	−31	
(*f*)	$17x - 21y + 38z = 89$	1		
(*g*)	$-16x - 17y + 31z = 43$		19	
(*h*)	$55x + 5y = 65$	16		
(*i*)	$285x + 80y = 445$	−1		

(j) $595x = 595;$ $\therefore x = 1;$
$5y = 65 - 55x = 65 - 55 = 10;$ $\therefore y = 2;$
$19z = 12 + 19x + 13y = 12 + 19 + 26 = 57;$ $\therefore z = 3;$
$2w = 3 - 4x - 3y + 5z = 3 - 4 - 6 + 15 = 8;$ $\therefore w = 4.$

Here w is eliminated from (a) and (b), obtaining (e); from (a) and (c), obtaining (f); and from (a) and (d), obtaining (g). Then z is eliminated from (e) and (f), obtaining (h), and from (e) and (g), obtaining (i). Then y is eliminated from (h) and (i), obtaining (j), which contains only the single variable x. Hence $x = 1$. Now substituting this value of x in either (h) or (i), y is found; substituting these values of x and y in either (e), (f), or (g), z is found; and so on. (Solution by determinants, see p. 123.)

Approximate Solution of a Set of Simultaneous Equations of the First Degree When the Number of Equations is Greater Than the Number of Unknowns. (Method of Least Squares.)

Case 1. Single Unknown Quantity. Given n equations in one unknown x; for example, n equally careful, independent measurements of some physical quantity:

$$x = x_1,\ x = x_2,\ \ldots\ x = x_n.$$

As the "best" value of x, take the arithmetic mean, x_0, of the several determinations, namely, $x_0 = (x_1 + x_2 + \ldots + x_n)/n$. The quantities $v_1 = x_0 - x_1$, $v_2 = x_0 - x_2$, $\ldots$ $v_n = x_0 - x_n$ are called the **residuals** of the observed values with respect to x_0, and their absolute values (that is, their numerical values without regard to sign) are denoted by $|v_1|, |v_2|, \ldots |v_n|$. [It can be shown that the sum of the squares of the residuals with respect to x_0 is smaller than the sum of the squares of the residuals with respect to any other value x'_0; hence the name of the method: "least squares."]

The quantities r and r_0, defined exactly by Bessel's formulæ:

$$r = \frac{0.6745}{\sqrt{n-1}}\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2},$$

$$r_0 = \frac{0.6745}{\sqrt{n(n-1)}}\sqrt{v_1^2 + v_2^2 + \ldots + v_n^2},$$

or given approximately by the simpler formulæ of Peters:

$$r = \frac{0.8453}{\sqrt{n(n-1)}}(|v_1| + |v_2| + \ldots + |v_n|),$$

$$r_0 = \frac{0.8453}{n\sqrt{n-1}}(|v_1| + |v_2| + \ldots + |v_n|),$$

are called the **probable error of a single observation** (r), and the **probable error of the mean** (r_0), for the given series of observations. Note that $r_0 = r/\sqrt{n}$. For tables of the coefficients, see p. 63. This quantity r (or r_0) is best regarded as merely a conventional means of recording the relative precision of different sets of observations. If r is small, it may be inferred that most errors of the "accidental" class have been eliminated; but it should be especially noted that the smallness of r gives no information in regard to "constant" or "systematic" errors.

A statement like "x is equal to 2.36 with a probable error of 0.02," is written: $x = 2.36 \pm 0.02$, and is usually understood to mean that the true value of x, as far as can be told, is just as likely to lie inside as outside the interval from 2.34 to 2.38.

To test the **distribution of residuals**, arrange the residuals in order of magnitude, without regard to sign, and count the number, y, of residuals which are numerically less than some assigned value a; divide y by n, the total number of observations, and divide a by r, the probable error of a single observation. Do this for various values of a, and compare the results with the table on p. 63, which gives the standard distribution of residuals, as found from experience from a large number of different series of observations. In particular, the number of residuals numerically less than r should be about equal to the number numerically greater than r (if n is large). If any large discrepancy appears, the series of observations should be regarded as unsatisfactory.

NOTE. The "mean square error" sometimes met with is equal to the probable error divided by 0.6745.

Case 2. Several Unknown Quantities. Assume that there have been obtained by measurement or observation n different equations of the first degree involving, say, three unknown quantities, x, y, z. There are then n simultaneous equations in three unknowns, and if $n > 3$ there will be, in general, no set of values of x, y, z which will satisfy all these n equations exactly. In such a case, the "best" set of values, x_0, y_0, z_0, may be found by the method of least squares as follows. (The process usually involves a large amount of labor; the use of a computing machine is advisable.)

Given Equations

$$\begin{aligned} a_1x + b_1y + c_1z &= p_1 \\ a_2x + b_2y + c_2z &= p_2 \\ \cdot \quad \cdot \quad & \cdot \quad \cdot \\ a_nx + b_ny + c_nz &= p_n \end{aligned}$$

First, arrange the n given equations in the form indicated, being careful not to modify any of them by multiplication or division. (Any of the coefficients may of course be zero.)

Next, form the three "normal equations" as follows: (1) Multiply each of the given equations by the coefficient of x in that equation, and add; the result will be the first normal equation. (2) Multiply each of the given equations by the coefficient of y in that equation, and add; the result will be the second normal equation. (3) Similarly for the third. {Notation: $[aa] = a_1^2 + a_2^2 + \ . \ . \ . + a_n^2$; $[ab] = a_1b_1 + a_2b_2 + \ . \ . \ . + a_nb_n$; $[ap] = a_1p_1 + a_2p_2 + \ . \ . \ . + a_np_n$; etc.}

Normal Equations

$$\begin{aligned} [aa]x_0 + [ab]y_0 + [ac]z_0 &= [ap] \\ [ba]x_0 + [bb]y_0 + [bc]z_0 &= [bp] \\ [ca]x_0 + [cb]y_0 + [cc]z_0 &= [cp] \end{aligned}$$

Finally, solve the three normal equations for the three unknowns in the usual way.

The quantities $v_1 = a_1x_0 + b_1y_0 + c_1z_0 - p_1$, etc., are called the **residuals** with respect to x_0, y_0, z_0. [It can be shown that the sum of the squares of the residuals with respect to x_0, y_0, z_0 is smaller than the corresponding quantity with respect to any other set of values, x'_0, y'_0, z'_0; this relation is taken as the criterion for the "best" set of values of x, y, z.]

The **probable error of a single observation** is

$$r = \frac{0.6745}{\sqrt{n-m}} \sqrt{v_1^2 + v_2^2 + \ . \ . \ . + v_n^2}, \text{ or approximately,}$$

$$r = \frac{0.8453}{\sqrt{n(n-m)}} (|v_1| + |v_2| + \ . \ . \ . + |v_n|),$$

where m = the number of unknown quantities (here $m = 3$).

DETERMINANTS

Determinants are used chiefly in formulating theoretical results; they are seldom of use in numerical computation.

Evaluation of Determinants:

Of the second order:

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Of the third order:

$$\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 c_2 \\ b_3 c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 c_1 \\ b_3 c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 c_1 \\ b_2 c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Of the fourth order:

$$\begin{vmatrix} a_1 b_1 c_1 d_1 \\ a_2 b_2 c_2 d_2 \\ a_3 b_3 c_3 d_3 \\ a_4 b_4 c_4 d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 c_2 d_2 \\ b_3 c_3 d_3 \\ b_4 c_4 d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 c_1 d_1 \\ b_3 c_3 d_3 \\ b_4 c_4 d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 c_1 d_1 \\ b_2 c_2 d_2 \\ b_4 c_4 d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 c_1 d_1 \\ b_2 c_2 d_2 \\ b_3 c_3 d_3 \end{vmatrix}$$

etc. In general, to evaluate a determinant of the nth order, take the elements of the first column with signs alternately plus and minus, and form the sum of the products obtained by multiplying each of these elements by its corresponding **minor.** The minor corresponding to any element a_1 is the determinant (of next lower order) obtained by striking out from the given determinant the row and column containing a_1.

Properties of Determinants.

1. The columns may be changed to rows and the rows to columns:

$$\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix} = \begin{vmatrix} a_1 a_2 a_3 \\ b_1 b_2 b_3 \\ c_1 c_2 c_3 \end{vmatrix}$$

2. Interchanging two columns changes the sign of the result.
3. If two columns are equal, the determinant is zero.
4. If the elements of one column are m times the elements of another column, the determinant is zero.
5. To multiply a determinant by any number m, multiply all the elements of any one column by m.
6. $$\begin{vmatrix} a_1 + p_1 + q_1, & b_1 & c_1 \\ a_2 + p_2 + q_2, & b_2 & c_2 \\ a_3 + p_3 + q_3, & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix} + \begin{vmatrix} p_1 b_1 c_1 \\ p_2 b_2 c_2 \\ p_3 b_3 c_3 \end{vmatrix} + \begin{vmatrix} q_1 b_1 c_1 \\ q_2 b_2 c_2 \\ q_3 b_3 c_3 \end{vmatrix}$$
7. $$\begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix} = \begin{vmatrix} a_1 + mb_1, & b_1 & c_1 \\ a_2 + mb_2, & b_2 & c_2 \\ a_3 + mb_3, & b_3 & c_3 \end{vmatrix}$$

Solution of Simultaneous Equations by Determinants.

If
$$\begin{aligned} a_1x + b_1y + c_1z &= \mathrm{p}_1 \\ a_2x + b_2y + c_2z &= \mathrm{p}_2 \\ a_3x + b_3y + c_3z &= \mathrm{p}_3 \end{aligned} \qquad \text{where } D = \begin{vmatrix} a_1 b_1 c_1 \\ a_2 b_2 c_2 \\ a_3 b_3 c_3 \end{vmatrix} \neq 0,$$

then
$$\begin{aligned} x &= D_1/D, \\ y &= D_2/D, \text{ where } D_1 = \begin{vmatrix} \mathrm{p}_1 b_1 c_1 \\ \mathrm{p}_2 b_2 c_2 \\ \mathrm{p}_3 b_3 c_3 \end{vmatrix}, \ D_2 = \begin{vmatrix} a_1 \mathrm{p}_1 c_1 \\ a_2 \mathrm{p}_2 c_2 \\ a_3 \mathrm{p}_3 c_3 \end{vmatrix}, \ D_3 = \begin{vmatrix} a_1 b_1 \mathrm{p}_1 \\ a_2 b_2 \mathrm{p}_2 \\ a_3 b_3 \mathrm{p}_3 \end{vmatrix} \\ z &= D_3/D, \end{aligned}$$

Similarly for a larger (or smaller) number of equations.

THE ALGEBRA OF IMAGINARY OR COMPLEX QUANTITIES

In the algebra of imaginary or complex quantities, the objects on which the operations of the algebra are performed are not numbers in any ordinary sense of the word, but are best thought of as **points in a plane** (or as **vectors** drawn from a fixed origin to these points). The "**complex plane**" is determined by three fundamental points, O, U, i, arranged as in Fig. 2 and called the **zero point**, the **unit point**, and the **imaginary unit point**, respectively. All points on the line through O and U are called **real points**—positive if on the right of O, negative if on the left. All the remaining points in the plane are called **imaginary points**—those on the line through O and i being called the **pure imaginary points.**

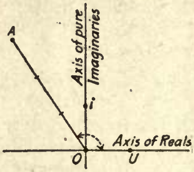


FIG. 2.

The position of any point A in the plane may be determined by the *distance* from the origin O, measured in terms of OU as the unit length, and the *angle* φ which OA makes with the positive direction of the axis of reals. The distance r is sometimes called the modulus or absolute value of the point; the angle φ is sometimes called the amplitude or argument of the point. The notation $A = (3, \angle 120°)$ means the point whose *distance*, r, is 3 times OU, and whose *angle*, φ, is 120°. The development of the algebra depends wholly on the definitions of three fundamental operations denoted by $A + B$, $A \times B$, and e^A, as follows.

Addition and Subtraction. The **sum,** $A+B$, of two points A and B is defined as the point reached by starting from A and performing a journey equal in length and direction to the journey from O to B. That is, the vector from O to $A + B$ is the vector sum of the vectors OA and OB. In case A and B are not in line with O, the point $A + B$ is the fourth vertex of a parallelogram of which OA and OB are the sides (Fig. 3). Conversely, if any two points A and B are given, there is a definite point X such that $A = B + X$; this point X is called the **remainder,** A minus B, and is denoted by $A - B$. The point $O - B$ is denoted for brevity by $-B$. With these definitions of $A + B$ and $A - B$, all the ordinary laws of addition and subtraction that hold in the algebra of real numbers hold also in the algebra of complex quantities. In particular, the zero point O has all the formal properties of the number zero, and is denoted by 0.

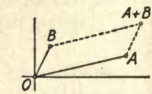


FIG. 3.

[Note: If A and B are "real" points, $A + B$ and $A - B$ will also be real.

Repeated Addition. Multiples and Submultiples. The point $A + A + A + \ldots + A$ to n terms is called the **nth multiple of A** and is denoted by nA. The points U, $2U$, $3U$, . . . are denoted, for brevity, by 1, 2, 3, ... Conversely, if any point A, and any positive integer n are given, there is a definite point X such that $nX = A$; this point X is called the **nth submultiple of A,** and is denoted by A/n. The points $U/2$, $U/3$, . . . are denoted, for brevity, by ½, ⅓, ...

Multiplication and Division. The **product,** $A \times B$, or $A{\cdot}B$, or AB, of two points A and B is defined as the point whose angle is the sum of the angles of the given points, and whose distance is the product of the distances. (See Fig. 4.) Thus, if $A = (5, \angle 120°)$ and $B = (2, \angle 270°)$, then $AB = (10, \angle 30°)$. Conversely, if any two points A and B are given, provided B is not zero, there is a definite point X such that

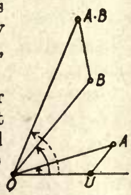


FIG. 4.

$A = BX$. This point X is called the **quotient**, A divided by B, and is denoted by A/B (where $B \neq 0$). Thus, the point A/B is a point whose angle is the angle of A minus the angle of B, and whose distance is the distance of A divided by the distance of B. The point U/B ($B \neq 0$) is called the **reciprocal** of the point B, and is denoted by $1/B$. (See Fig. 5.) With these definitions of AB and A/B the elementary laws of multiplication and division that hold in the algebra of real numbers hold also in the algebra of complex quantities. In particular, the point U has all the formal properties of the number unity, and is denoted by 1.

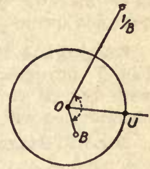


FIG. 5.

[Note: If A and B are real, AB and A/B will also be real.]

Repeated Multiplication. Powers and Roots. The point $A \times A \times A \times \ldots \times A$ to n factors is called the nth power of A and is denoted by A^n (Fig. 6). Conversely, if any point A (not 0) and any positive integer n are given, there will be n distinct points X such that $X^n = A$; each of these points is called an **nth root of A,** some one of them, usually the one with the smallest positive angle, being denoted by $\sqrt[n]{A}$ or $A^{1/n}$. Thus, the point $\sqrt[n]{A}$ is a point whose distance is the nth root of the distance of A, and whose angle is $1/n$th of the angle of A. All the nth roots of A will lie on the circumference of a circle about O as center, and will divide that circumference into n equal parts (Fig. 7). Every point A (not 0) has two square roots, three cube roots, etc. Hence the theorem "If $A^n = B^n$ then $A = B$" does not hold in this algebra, and the ordinary rules for radical signs must be applied with caution. For example, if A and B are positive reals, $\sqrt{-A}\cdot\sqrt{-B} = -\sqrt{AB}$ and not $\sqrt{(-A)(-B)}$, which would give $+\sqrt{AB}$.

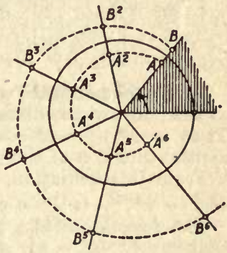


FIG. 6.

[Note: If A is real and positive, $\sqrt[n]{A}$ will be real and positive; if A is real and negative, $\sqrt[n]{A}$ will be real if n is odd and imaginary if n is even.]

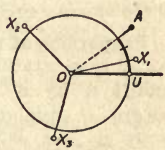


FIG. 7.

Properties of i. The point i is the point whose distance is 1 and whose angle is 90 deg. It follows from the definition above that **multiplying any point A by i has the effect of rotating the point through an angle of + 90°** without changing its distance from O. In particular,

$i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.; $i = \sqrt{-1}$, $-i = -\sqrt{-1}$; where "1" denotes not the *number* one, but the *point* U.

Similarly, multiplying any point A by -1 has the effect of rotating the point through 180 deg.

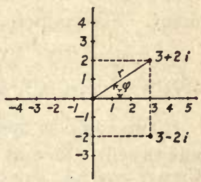


FIG. 8.

First Standard Form for a Complex Quantity (Fig. 8). Any point A can be expressed in the form $x + iy$, where x and y are real points. For example, the three cube roots of 1 are 1, $-\frac{1}{2} + \frac{1}{2}i\sqrt{3}$, and $-\frac{1}{2} - \frac{1}{2}i\sqrt{3}$.

In general, $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$;

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2);$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}.$$

If two complex quantities are equal, their real parts must be equal, and the coefficients of their pure imaginary parts must also be equal. That is, if $x_1 + iy_1 = x_2 + iy_2$, then $x_1 = x_2$ and $y_1 = y_2$. Thus a single equation between complex quantities is equivalent to two equations between real quantities.

Conjugate Imaginaries. Two points $A = x + iy$ and $B = x - iy$ are called conjugate imaginaries. Two such points are symmetrically situated with regard to the axis of reals. The sum and product of two conjugate imaginaries will be real.

Second Standard Form for a Complex Quantity. Since $x = r\cos\varphi$ and $y = r\sin\varphi$, any point $A = x + iy$ can be expressed $A = r(\cos\varphi + i\sin\varphi)$, where r is real and positive (namely, the distance of A), and φ is real (namely the angle of A). For example, the three cube roots of 1 are 1, $\cos 120° + i\sin 120°$, and $\cos 240° + i\sin 240°$. In general,
$[r_1(\cos\varphi_1 + i\sin\varphi_1)][r_2(\cos\varphi_2 + i\sin\varphi_2)] = r_1r_2[(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)]$;
$[r(\cos\varphi + i\sin\varphi)]^n = r^n[\cos(n\varphi) + i\sin(n\varphi)]$ (**De Moivre's Theorem**).

The Exponential Function, e^A, or $\exp A$, of any point $A = x + iy$ is defined as the point whose distance is e^x and whose angle (measured in radians) is y. That is, $e^{x+iy} = e^x(\cos y + i\sin y)$. Here e^x means the ordinary exponential function of the real quantity x, where $e = 2.718$.

From this definition, the usual formal laws of exponents can be deduced: $e^Ae^B = e^{A+B}$, $(e^A)^n = e^{nA}$, $e^{-A} = 1/e^A$; $e^1 = e$, $e^0 = 1$.

The function e^A is a periodic function with a pure imaginary period $2\pi i$; that is, $e^{A \pm k2\pi i} = e^A$, where k is any positive integer.

If A is made to move along a line parallel to the axis of reals [or axis of pure imaginaries], the corresponding point e^A will move along a straight line through O [or along a circle about O as center].

Properties of $e^{i\varphi}$. The point $e^{i\varphi}$ is a point whose distance is 1 and whose angle is φ. It follows from the definitions above that **multiplying any point A by $e^{i\varphi}$ has the effect of rotating the point through an angle** φ, without changing its distance from O. In particular, $e^{i\pi} = -1$, $e^{-i\pi} = -1$; $e^{i\pi/2} = i$; $e^{-i\pi/2} = -i$; $e^{2\pi i} = 1$.

Third Standard Form for a Complex Quantity. Any point A can be expressed in the form $A = re^{i\varphi}$, where r is the distance and φ the angle of the point. For example, the three cube roots of 1 are 1, $e^{\frac{1}{3}2\pi i}$, $e^{\frac{2}{3}2\pi i}$. In general,

$$(r_1e^{i\varphi_1})(r_2e^{i\varphi_2}) = (r_1r_2)e^{i(\varphi_1+\varphi_2)}; \quad (re^{i\varphi})^n = (r^n)e^{in\varphi}.$$

If $x + iy = re^{i\varphi}$, then $r = \sqrt{x^2 + y^2}$, $\sin\varphi = \dfrac{y}{r}$, $\cos\varphi = \dfrac{x}{r}$, $\tan\varphi = \dfrac{y}{x}$.

If two complex quantities are equal, their distances will be equal, and their angles will differ at most by some multiple of 2π. Thus, if $r_1e^{i\varphi_1} = r_2e^{i\varphi_2}$ then $r_1 = r_2$ and $\varphi_1 = \varphi_2$ or $\varphi_2 \pm k2\pi$. Here again a single equation between complex quantities is equivalent to two equations between real quantities.

Definition of $\mathbf{A}^B$. Let $A = re^{i\varphi}$; then $A^B = \exp[(\log_e r + i\varphi)B]$.

For example, $i^i = e^{-\pi/2}$ where $i = \sqrt{-1}$.

If a is a positive real, $a^{x+iy} = a^x [\cos (y \log_e a) + i \sin (y \log_e a)]$.

Trigonometric and Hyperbolic Functions of a Complex Variable. If A is any point, then, by definition,

$$\sin A = \frac{e^{iA} - e^{-iA}}{2i}, \quad \cos A = \frac{e^{iA} + e^{-iA}}{2}, \quad \tan A = \frac{\sin A}{\cos A} \quad (\cos A \neq 0);$$

$$\sinh A = \frac{e^{A} - e^{-A}}{2}, \quad \cosh A = \frac{e^{A} + e^{-A}}{2}, \quad \tanh A = \frac{\sinh A}{\cosh A}.$$

Hence the formulæ that hold for these functions in the real case (p. 131; p. 135; p. 161) hold also for the complex case. Further:

$$\begin{aligned} \sin (x + iy) &= \sin x \cosh y + i \cos x \sinh y, & \sin iy &= i \sinh y; \\ \cos (x + iy) &= \cos x \cosh y - i \sin x \sinh y, & \cos iy &= \cosh y; \\ \sinh (x + iy) &= \sinh x \cos y + i \cosh x \sin y, & \sinh iy &= i \sin y; \\ \cosh (x + iy) &= \cosh x \cos y + i \sinh x \sin y, & \cosh iy &= \cos y; \end{aligned}$$

where $\sin x$, $\sinh x$, etc., are the ordinary trigonometric and hyperbolic functions of the real variables x and y. The functions $\sin A$ and $\cos A$ are periodic with a real period 2π. The functions $\sinh A$ and $\cosh A$ are periodic with a pure imaginary period $2\pi i$.

Logarithmic and Other Inverse Functions of a Complex Variable. If any point A is given, there will be an infinite number of points X such that $e^X = A$; any one of these points may be called a logarithm of A, and be denoted by $\log A$. All the values of the logarithm of A may be obtained from any one value by adding multiples of $2\pi i$.

If $x + iy = re^{i\varphi}$, then $\log_e (x + iy) = \log_e r + i\varphi \pm k \cdot 2\pi i$.

If any point A is given, there will be an infinite number of points X such that $\sin X = A$; any one of these may be denoted by $\sin^{-1} A$. The functions $\cos^{-1} A$, $\sinh^{-1} A$, etc., are defined in a similar way.

The elementary laws of operation which hold for these functions in the algebra of reals hold also, in a general way, in the algebra of complex quantities; but caution must be used, on account of the ambiguity in the symbols $\log A$, $\sin^{-1} A$, etc., which denote many-valued functions.

Differentiation of Functions of a Complex Variable. If $w = f(z)$, the derivative of w with respect to z is defined as

$$dw/dz = \lim \{[f(z + \Delta z) - f(z)]/\Delta z\} \text{ when } \Delta z \text{ approaches } 0.$$

It can be shown that $\lim \{[\exp \Delta z - 1]/\Delta z\} = 1$; hence $d(e^z) = e^z dz$, $d(\sin z) = \cos z \, dz$, etc., so that the formulæ for differentiation here are the same as in the case of a real variable (p. 157).

NOTE. For the algebra of vector analysis, which differs in important respects from the algebra of complex quantities, see p. 185.

TRIGONOMETRY

FORMAL TRIGONOMETRY

Angles, or Rotations. An **angle** is generated by the rotation of a ray, as Ox, about a fixed point O in the plane. Every angle has an **initial line** (OA) from which the rotation started (Fig. 1), and a **terminal line** (OB) where it stopped; and the counterclockwise direction of rotation is taken as positive. Since the rotating ray may revolve as often as desired, angles of any magnitude, positive or negative, may be obtained. Two angles are **congruent** if they may be superposed so that their initial lines coincide and their terminal lines coincide. That is, two congruent angles are either equal or differ by some multiple of 360 deg. Two angles are **complementary** if their sum is 90 deg.; **supplementary** if their sum is 180 deg. (The acute angles of a right-angled triangle are complementary.) If the initial line is placed so that it runs horizontally to the right, as in Fig. 2, then the angle is said to be an angle in the 1st, 2nd, 3rd, or 4th **quadrant** according as the terminal line lies across the region marked I, II, III, or IV. The angles 0 deg., 90 deg., 180 deg., 270 deg. are called the quadrantal angles.

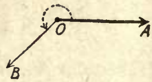


FIG. 1.

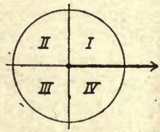


FIG. 2.

Units of Angular Measurement.

(1) SEXAGESIMAL MEASURE. (360 degrees = 1 revolution.) 1 degree = $1° = 1/90$ of a right angle. The degree is usually divided into 60 equal parts called minutes ($'$), and each minute into 60 equal parts called seconds ($''$); while the second is subdivided decimally. But for many purposes it is more convenient to divide the degree itself into decimal parts, thus avoiding the use of minutes and seconds. (See tables, pp. 46–51.)

(2) CENTESIMAL MEASURE, used chiefly in France. (400 grades = 1 revolution.) 1 grade = $1/100$ of a right angle. The grade is always divided decimally, the following terms being sometimes used: 1 "centesimal minute" = $1/100$ of a grade; 1 "centesimal second" = $1/100$ of a centesimal minute. In reading Continental books it is important to notice carefully which system is employed.

(3) RADIAN, OR CIRCULAR, MEASURE. (π radians = 180 degrees.) 1 radian = the angle subtended by an arc whose length is equal to the length of the radius. The radian is constantly used in higher mathematics and in mechanics, and is always divided decimally. Table, pp. 44–45.

1 radian $= 57°.30- = 57°.2957795131 = 57°\,17'\,44''.806247 = 180°/\pi$.

$1° = 0.01745\ldots$ radian $= 0.01745\,32925$ radian.

$1' = 0.00029\,08882$ radian. $1'' = 0.00000\,48481$ radian.

(For 10-place conversion tables, see the Smithsonian Tables of Hyperbolic Functions, Washington, D. C.)

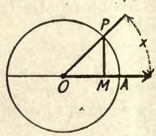


FIG. 3.

Definitions of the Trigonometric Functions. Let x be any angle whose initial line is OA and terminal line OP (see Fig. 3). Drop a perpendicular from P on OA or OA produced. In the right triangle OMP, the three sides are MP = "side opposite" O (positive if running upward); OM = "side adjacent" to O (positive if running to the right); OP = "hypothenuse" or "radius" (may always be taken as positive); and the six ratios between these sides are the principal trigonometric functions of the angle x; thus:

$$
\begin{aligned}
\text{sine of } x &= \sin x = \text{opp/hyp} = MP/OP;\\
\text{cosine of } x &= \cos x = \text{adj/hyp} = OM/OP;\\
\text{tangent of } x &= \tan x = \text{opp/adj} = MP/OM;\\
\text{cotangent of } x &= \cot x = \text{adj/opp} = OM/MP;\\
\text{secant of } x &= \sec x = \text{hyp/adj} = OP/OM;\\
\text{cosecant of } x &= \csc x = \text{hyp/opp} = OP/MP.
\end{aligned}
$$

The last three are best remembered as the reciprocals of the first three:

$$\cot x = 1/\tan x;\ \sec x = 1/\cos x;\ \csc x = 1/\sin x.$$

Other functions in use are the versed sine, the coversed sine, and the exterior secant:

$$\text{vers } x = 1 - \cos x;\ \text{covers } x = 1 - \sin x;\ \text{exsec } x = \sec x - 1.$$

For graphs, see p. 174; series, p. 161.

Signs of the Trigonometric Functions

If x is in quadrant	I	II	III	IV
sin x and csc x are..................	+	+	−	−
cos x and sec x are..................	+	−	−	+
tan x and cot x are..................	+	−	+	−

vers x and covers x are always positive.

Variations in the Functions as x Varies from 0 deg. to 360 deg. are shown in the accompanying table. The variations in the sine and cosine are

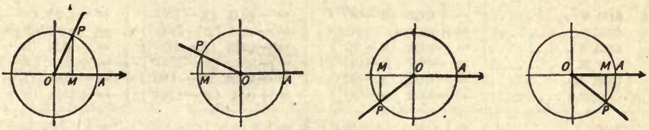


FIG. 4.

best remembered by noting the changes in the lines MP and OM (Fig. 4) in the "unit circle" (that is, a circle with radius $= OP = 1$), as P moves around the circumference.

x	0° to 90°	90° to 180°	180° to 270°	270° to 360°	Values at		
					30°	45°	60°
sin x	+0 to +1	+1 to +0	−0 to −1	−1 to −0	½	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
csc x	+∞ to +1	+1 to +∞	−∞ to −1	−1 to −∞	2	$\sqrt{2}$	$\frac{2}{3}\sqrt{3}$
cos x	+1 to +0	−0 to −1	−1 to −0	+0 to +1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	½
sec x	+1 to +∞	−∞ to −1	−1 to −∞	+∞ to +1	$\frac{2}{3}\sqrt{3}$	$\sqrt{2}$	2
tan x	+0 to +∞	−∞ to −0	+0 to +∞	−∞ to −0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$
cot x	+∞ to +0	−0 to −∞	+∞ to +0	−0 to −∞	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$
vers x	+0 to +1	+1 to +2	+2 to +1	+1 to +0			
covers x	+1 to +0	+0 to +1	+1 to +2	+2 to +1			

$\sqrt{2} = 1.4142$; $\frac{1}{2}\sqrt{2} = 0.7071$; $\sqrt{3} = 1.7321$; $\frac{1}{2}\sqrt{3} = 0.8660$; $\frac{1}{3}\sqrt{3} = 0.5774$; $\frac{2}{3}\sqrt{3} = 1.1547$

Trigonometrical Tables. The tables on pp. 46–56 give the values of the principal trigonometric functions and of their logarithms, correct to four places of decimals, the angle advancing either by tenths of a degree (p. 46) or by 10 min. (p. 52). These tables will be found adequate for most

computations in which an accuracy of 1 part in 1000 is sufficient. If much computing is to be done, it is advisable to use a separate volume of tables, containing more facilities for interpolation, and printed in larger type, such as the four-place tables of E. V. Huntington (Harvard Coöperative Society, Cambridge, Mass.), with convenient marginal tabs; the five-place tables published by Macmillan or many others; the six-place tables of Bremiker; the standard seven-place tables of Schrön, Vega, or Bruhns (angles advancing by 10 sec.); or the great eight-place of Bauschinger and Peters (angles advancing at intervals of 1 sec. from 0 deg. to 90 deg.). The larger tables give only the logarithms of the functions, not the natural values.

To Find Any Function of a Given Angle. (Reduction to the first quadrant.) It is often required to find the functions of any angle x from a table that includes only angles between 0 deg. and 90 deg. If x is not already between 0 deg. and 360 deg., first "reduce to the first revolution" by simply adding or subtracting the proper multiple of 360 deg.; [for any function of (x) = the same function of $(x \pm n \times 360°)$]. Next **reduce to the first quadrant** as follows:

If x is between	90° and 180°	180° and 270°	270° and 360°
Subtract	90° from x	180° from x	270° from x
Then **sin x**..........	$= +\mathbf{cos}\ (x-90°)$	$= -\mathbf{sin}\ (x-180°)$	$= -\mathbf{cos}\ (x-270°)$
csc x..........	$= +\sec\ (x-90°)$	$= -\csc\ (x-180°)$	$= -\sec\ (x-270°)$
cos x..........	$= -\mathbf{sin}\ (x-90°)$	$= -\mathbf{cos}\ (x-180°)$	$= +\mathbf{sin}\ (x-270°)$
sec x..........	$= -\csc\ (x-90°)$	$= -\sec\ (x-180°)$	$= +\csc\ (x-270°)$
tan x..........	$= -\mathbf{cot}\ (x-90°)$	$= +\mathbf{tan}\ (x-180°)$	$= -\mathbf{cot}\ (x-270°)$
cot x..........	$= -\tan\ (x-90°)$	$= +\text{oot}\ (x-180°)$	$= -\tan\ (x-270°)$
vers x..........	$= 1+\sin\ (x-90°)$	$= 1+\cos\ (x-180°)$	$= 1-\sin\ (x-270°)$
covers x.......	$= 1-\cos\ (x-90°)$	$= 1+\sin\ (x-180°)$	$= 1+\cos\ (x-270°)$

The "reduced angle" ($x - 90°$, or $x - 180°$, or $x - 270°$) will in each case be an angle between 0° and 90°, whose functions can then be found in the table.

[Note. The formulæ for sine and cosine are best remembered by aid of the unit circle.]

To Find the Angle When One of Its Functions is Given. In general, there will be two angles between 0 deg. and 360 deg. corresponding to any given function. The following tabulated rules show how to find these angles.

Given	First find from the tables an *acute* angle x_0 such that	Then the required angles x_1 and x_2 will be
$\sin x = +a$	$\sin x_0 = a$	x_0 and $180° - x_0$
$\cos x = +a$	$\cos x_0 = a$	x_0 and $[360° - x_0]$
$\tan x = +a$	$\tan x_0 = a$	x_0 and $[180° + x_0]$
$\cot x = +a$	$\cot x_0 = a$	x_0 and $[180° + x_0]$
$\sin x = -a$	$\sin x_0 = a$	$[180° + x_0]$ and $[360° - x_0]$
$\cos x = -a$	$\cos x_0 = a$	$180° - x_0$ and $[180° + x_0]$
$\tan x = -a$	$\tan x_0 = a$	$180° - x_0$ and $[360° - x_0]$
$\cot x = -a$	$\cot x_0 = a$	$180° - x_0$ and $[360° - x_0]$

The angles enclosed in brackets lie outside the range from 0 deg. to 180 deg., and hence cannot occur as angles in a triangle.

For solution of trigonometric equations, see p. 118.

Relations Between the Functions of a Single Angle. (See Fig. 5.)

$$\sin^2 x + \cos^2 x = 1;\ \tan x = \frac{\sin x}{\cos x};\ \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x};$$

$$1 + \tan^2 x = \sec^2 x = \frac{1}{\cos^2 x};\ 1 + \cot^2 x = \csc^2 x = \frac{1}{\sin^2 x};$$

$$\sin x = \sqrt{1 - \cos^2 x} = \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \frac{1}{\sqrt{1 + \cot^2 x}};$$

$$\cos x = \sqrt{1 - \sin^2 x} = \frac{1}{\sqrt{1 + \tan^2 x}} = \frac{\cot x}{\sqrt{1 + \cot^2 x}}.$$

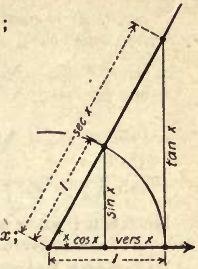


FIG. 5.

Functions of Negative Angles. $\sin(-x) = -\sin x$; $\cos(-x) = \cos x$; $\tan(-x) = -\tan x$.

Functions of the Sum and Difference of Two Angles.

$$\sin(x + y) = \sin x \cos y + \cos x \sin y;$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y;$$
$$\tan(x + y) = [\tan x + \tan y]/[1 - \tan x \tan y];$$
$$\cot(x + y) = [\cot x \cot y - 1]/[\cot x + \cot y];$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y;$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y;$$
$$\tan(x - y) = [\tan x - \tan y]/[1 + \tan x \tan y];$$
$$\cot(x - y) = [\cot x \cot y + 1]/[\cot y - \cot x];$$

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y);$$
$$\sin x - \sin y = 2 \cos \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y);$$
$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y);$$
$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y);$$

$$\tan x + \tan y = \frac{\sin(x + y)}{\cos x \cos y};\quad \cot x + \cot y = \frac{\sin(x + y)}{\sin x \sin y};$$

$$\tan x - \tan y = \frac{\sin(x - y)}{\cos x \cos y};\quad \cot x - \cot y = \frac{\sin(y - x)}{\sin x \sin y};$$

$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x + y)\sin(x - y);$$
$$\cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x = \cos(x + y)\cos(x - y);$$
$$\sin(45° + x) = \cos(45° - x);\ \tan(45° + x) = \cot(45° - x);$$
$$\sin(45° - x) = \cos(45° + x);\ \tan(45° - x) = \cot(45° + x).$$

In the following transformations, a and b are supposed to be positive, $c = \sqrt{a^2 + b^2}$, A = the positive acute angle for which $\tan A = a/b$, and B = the positive acute angle for which $\tan B = b/a$:

$$a \cos x + b \sin x = c \sin(A + x) = c \cos(B \mp x);$$
$$a \cos x - b \sin x = c \sin(A - x) = c \cos(B + x).$$

Functions of Multiple Angles and Half Angles.

$$\sin 2x = 2 \sin x \cos x;\ \sin x = 2 \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x;$$
$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1;$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x};\ \cot 2x = \frac{\cot^2 x - 1}{2 \cot x};$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x;\ \tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x};$$

$$\cos 3x = 4 \cos^3 x - 3 \cos x;$$

$$\sin(nx) = n \sin x \cos^{n-1} x - (n)_3 \sin^3 x \cos^{n-3} x + (n)_5 \sin^5 x \cos^{n-5} x - \ldots;$$

$$\cos(nx) = \cos^n x - (n)_2 \sin^2 x \cos^{n-2} x + (n)_4 \sin^4 x \cos^{n-4} x - \ldots,$$

where $(n)_2$, $(n)_3$, . . . are the binomial coefficients (see p. 39).

$$\sin \tfrac{1}{2} x = \pm \sqrt{\tfrac{1}{2}(1 - \cos x)};\ 1 - \cos x = 2 \sin^2 \tfrac{1}{2}x;$$

$$\cos \tfrac{1}{2}x = \pm \sqrt{\tfrac{1}{2}(1 + \cos x)};\ 1 + \cos x = 2 \cos^2 \tfrac{1}{2}x;$$

$$\tan \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x};$$

$$\tan\left(\frac{x}{2} + 45^\circ\right) = \pm \sqrt{\frac{1 + \sin x}{1 - \sin x}}.$$

Here the + or − sign is to be used according to the sign of the left-hand side of the equation.

Relations Between Three Angles Whose Sum is 180°.

$$\sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \cos \tfrac{1}{2}C;$$
$$\cos A + \cos B + \cos C = 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C + 1;$$
$$\sin A + \sin B - \sin C = 4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \cos \tfrac{1}{2}C;$$
$$\cos A + \cos B - \cos C = 4 \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B \sin \tfrac{1}{2}C - 1;$$
$$\sin^2 A + \sin^2 B + \sin^2 C = 2 \cos A \cos B \cos C + 2;$$
$$\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C;$$
$$\tan A + \tan B + \tan C = \tan A \tan B \tan C;$$
$$\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C = \cot \tfrac{1}{2}A \cot \tfrac{1}{2}B \cot \tfrac{1}{2}C;$$
$$\cot A \cot B + \cot A \cot C + \cot B \cot C = 1;$$
$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C;$$
$$\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C.$$

Inverse Trigonometric Functions. The notation $\sin^{-1}x$ (read: antisine of x, or inverse sine of x; sometimes written arc sin x) means the principal angle whose sine is x. Similarly for $\cos^{-1}x$, $\tan^{-1}x$, etc. (The principal angle means an angle between −90° and +90° in case of $\sin^{-1}$ and $\tan^{-1}$, and between 0° and 180° in the case of $\cos^{-1}$.) For graphs, see p. 174.

SOLUTION OF PLANE TRIANGLES

The "parts" of a plane triangle are its three sides, a, b, c, and its three angles A, B, C (A being opposite a, B opposite b, C opposite c, and $A + B + C = 180°$). A triangle is, in general, determined by any three parts (not all angles). To "solve" a triangle means to find the unknown parts from the known. The fundamental formulæ are:

Law of sines: $\dfrac{a}{b} = \dfrac{\sin A}{\sin B}$. Law of cosines: $c^2 = a^2 + b^2 - 2ab \cos C$.

Right Triangles. Use the definitions of the trigonometric functions, selecting for each unknown part a relation which connects that unknown with known quantities; then solve the resulting equations. Thus, in Fig. 6, if $C = 90°$, then $A + B = 90°$, $c^2 = a^2 + b^2$,

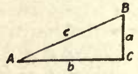


FIG. 6.

$$\sin A = a/c,\ \cos A = b/c,\ \tan A = a/b,\ \cot A = b/a.$$

If A is very small, use $\tan \tfrac{1}{2}A = \sqrt{(c - b)/(c + b)}$.

Oblique Triangles. There are four cases. It is highly desirable in all these cases to draw a sketch of the triangle approximately to scale before commencing the computation, so that any large numerical error may be readily detected.

Case 1. GIVEN TWO ANGLES (provided their sum is < 180 deg.), AND ONE

SIDE (say a, Fig. 7). The third angle is known, since $A + B + C = 180°$. To find the remaining sides, use $b = \dfrac{a \sin B}{\sin A}$, $c = \dfrac{a \sin C}{\sin A}$. Or, drop a perpendicular from either B or C on the opposite side, and solve by right triangles.

Check: $c \cos B + b \cos C = a$.

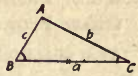


FIG. 7.

Case 2. GIVEN TWO SIDES (say a and b), AND THE INCLUDED ANGLE (C); and suppose $a > b$. Fig. 8.

First Method: Find c from $c^2 = a^2 + b^2 - 2ab \cos C$ [or $c^2 = (a - b)^2 + 2ab \text{ vers } C$]; then find the smaller angle, B, from $\sin B = (b/c) \sin C$; and finally, find A from $A = 180° - (B + C)$. Check: $a \cos B + b \cos A = c$.

Second Method: Find $\frac{1}{2}(A - B)$ from the law of tangents:

$$\tan \tfrac{1}{2}(A - B) = [(a - b)/(a + b)] \cot \tfrac{1}{2}C,$$

and $\frac{1}{2}(A + B)$ from $\frac{1}{2}(A + B) = 90° - C/2$; hence $A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B)$ and $B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B)$. Then find c from $c = a \sin C/\sin A$ or $c = b \sin C/\sin B$. Check: $a \cos B + b \cos A = c$.

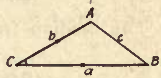


FIG. 8.

Third Method: Drop a perpendicular from A to the opposite side, and solve by right triangles.

Case 3. GIVEN THE THREE SIDES (provided the largest is less than the sum of the other two), Fig. 9.

First Method: Find the largest angle A (which may be acute or obtuse) from $\cos A = (b^2 + c^2 - a^2)/2bc$ $\{$or $\text{vers } A = [a^2 - (b - c)^2]/2bc\}$ and then find B and C (which will always be acute) from $\sin B = b \sin A/a$ and $\sin C = c \sin A/a$. Check: $A + B + C = 180°$.

Second Method: Find A, B, and C from $\tan \frac{1}{2}A = r/(s - a)$, $\tan \frac{1}{2}B = r/(s - b)$, $\tan \frac{1}{2}C = r/(s - c)$, where $s = \frac{1}{2}(a + b + c)$, and $r = \sqrt{(s - a)(s - b)(s - c)/s}$. Check: $A + B + C = 180°$.

Third Method: If only one angle, say A, is required, use

$$\sin \tfrac{1}{2}A = \sqrt{(s - b)(s - c)/bc} \quad \text{or}$$

$$\cos \tfrac{1}{2}A = \sqrt{s(s - a)/bc},$$

according as $\frac{1}{2}A$ is nearer 0° or nearer 90°.

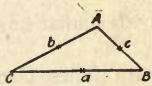


FIG. 9.

Case 4. GIVEN TWO SIDES (say b and c) AND THE ANGLE OPPOSITE ONE OF THEM (B). This is the "ambiguous case" in which there may be two solutions, or one, or none (see Fig. 10).

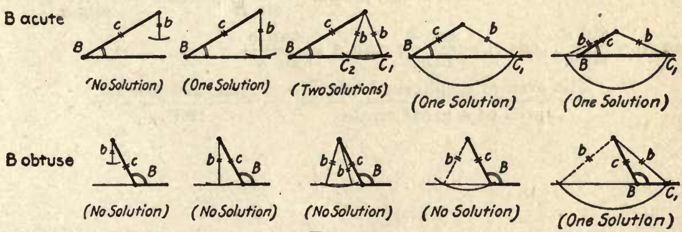


FIG. 10.

First, try to find C from $\sin C = c \sin B/b$. If $\sin C > 1$, there is no solution. If $\sin C = 1$, $C = 90°$ and the triangle is a right triangle. If $\sin C < 1$, this determines two angles C, namely, an acute angle C_1, and an obtuse angle $C_2 = 180° - C_1$. Then C_1 will yield a solution when and only when

$C_1 + B < 180°$ (see Case 1); and similarly C_2 will yield a solution when and only when $C_2 + B < 180°$ (see Case 1).

Other Properties of Triangles. (See also p. 99 and p. 105.)

Area $= \frac{1}{2}ab \sin C = \sqrt{s(s-a)(s-b)(s-c)} = rs$, where $s = \frac{1}{2}(a+b+c)$, and r = radius of inscribed circle $= \sqrt{(s-a)(s-b)(s-c)/s}$.

Radius of circumscribed circle $= R$, where

$$2R = a/\sin A = b/\sin B = c/\sin C;\ r = 4R \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{abc}{4Rs}.$$

The length of the bisector of the angle C is

$$z = \frac{2\sqrt{abs(s-c)}}{a+b} = \frac{\sqrt{ab[(a+b)^2 - c^2]}}{a+b}.$$

The median from C to the middle point of c is $m = \frac{1}{2}\sqrt{2(a^2+b^2) - c^2}$.

SOLUTION OF SPHERICAL TRIANGLES

For the occasional solution of a spherical triangle the following formulæ will be sufficient. For a detailed discussion, see any text-book on spherical trigonometry.

Let a, b, c be the sides of the spherical triangle, that is, portions of arcs of great circles of the sphere; and let A, B, C be the angles of the triangle, that is, the angles made by tangents drawn to the sides at their points of intersection on the sphere. The sum of the angles will always be greater than two right angles, and may be nearly six right angles. The angle $E = A + B + C - 180°$ is called the **spherical excess** of the triangle. (See also p. 100.)

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B};\quad \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C};\quad \frac{\sin c}{\sin C} = \frac{\sin a}{\sin A}.$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

with similar formulæ for $\cos b$ and $\cos c$.

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

with similar formulæ for $\cos B$ and $\cos C$.

In the special case of a right spherical triangle, in which $C = 90°$,

$$\cos c = \cos a \cos b = \cot A \cot B;\quad \cos a = \frac{\cos A}{\sin B};\quad \cos b = \frac{\cos B}{\sin A};$$

$$\sin A = \frac{\sin a}{\sin c};\quad \cos A = \frac{\tan b}{\tan c};\quad \tan A = \frac{\tan a}{\sin b}.$$

$$\frac{\text{The area of a spherical triangle}}{\text{area of a great circle}} = \frac{\text{spherical excess}}{720°}.$$

HYPERBOLIC FUNCTIONS

The **hyperbolic sine, hyperbolic cosine,** etc., of any number x, are functions of x which are closely related to the exponential e^x, and which have formal properties very similar to those of the trigonometric functions, sine, cosine, etc. Their definitions and fundamental properties are as follows (see also p. 127; graphs, p. 175; table, p. 60; series, p. 161):

$\sinh x = \frac{1}{2}(e^x - e^{-x})$; $\cosh x = \frac{1}{2}(e^x + e^{-x})$; $\tanh x = \sinh x/\cosh x$;

$\operatorname{csch} x = 1/\sinh x$; $\operatorname{sech} x = 1/\cosh x$; $\coth x = 1/\tanh x$;

$\cosh^2 x - \sinh^2 x = 1$; $1 - \tanh^2 x = \operatorname{sech}^2 x$; $1 - \coth^2 x = -\operatorname{csch}^2 x$;
$\sinh(-x) = -\sinh x$; $\cosh(-x) = \cosh x$; $\tanh(-x) = -\tanh x$;

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$;
$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$;
$\tanh(x \pm y) = (\tanh x \pm \tanh y)/(1 \pm \tanh x \tanh y)$;

$\sinh 2x = 2 \sinh x \cosh x$; $\cosh 2x = \cosh^2 x + \sinh^2 x$;
$\tanh 2x = (2 \tanh x)/(1 + \tanh^2 x)$;
$\sinh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x - 1)}$; $\cosh \frac{1}{2}x = \sqrt{\frac{1}{2}(\cosh x + 1)}$;
$\tanh \frac{1}{2}x = (\cosh x - 1)/(\sinh x) = (\sinh x)/(\cosh x + 1)$.

The **inverse hyperbolic sine** of y, denoted by $\sinh^{-1} y$, is the number whose hyperbolic sine is y; that is, the notation $x = \sinh^{-1} y$ means $\sinh x = y$. Similarly for $\cosh^{-1} y$, $\tanh^{-1} y$, etc. These functions are closely related to the logarithmic function, and are especially valuable in the integral calculus. For graphs, see p. 175.

$\sinh^{-1}(y/a) = \log_e(y + \sqrt{y^2 + a^2}) - \log_e a$;
$\cosh^{-1}(y/a) = \log_e(y + \sqrt{y^2 - a^2}) - \log_e a$;

$$\tanh^{-1}\frac{y}{a} = \frac{1}{2}\log_e \frac{a+y}{a-y};\quad \coth^{-1}\frac{y}{a} = \frac{1}{2}\log_e \frac{y+a}{y-a}.$$

The **anti-gudermannian** of an angle u, denoted by $\mathrm{gd}^{-1}u$, is a number defined by $\mathrm{gd}^{-1}u = \log_e \tan(\frac{1}{4}\pi + \frac{1}{2}u) = \int \sec u\, du$. When u is small, $\mathrm{gd}^{-1}u = u + \frac{1}{6}u^3 + \frac{1}{24}u^5 + \frac{61}{5040}u^7 + \ldots$.

ANALYTICAL GEOMETRY

THE POINT AND THE STRAIGHT LINE

Rectangular Co-ordinates (Fig. 1). Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$. Then, distance $P_1P_2 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$; slope of $P_1P_2 = m = \tan u = (y_2 - y_1)/(x_2 - x_1)$; co-ordinates of mid-point are $x = \frac{1}{2}(x_1 + x_2)$, $y = \frac{1}{2}(y_1 + y_2)$; co-ordinates of point $(1/n)$th of the way from P_1 to P_2 are $x = x_1 + (1/n)(x_2 - x_1)$, $y = y_1 + (1/n)(y_2 - y_1)$.

Let m_1, m_2 be the slopes of two lines; then, if the lines are parallel, $m_1 = m_2$; if the lines are perpendicular to each other, $m_1 = -1/m_2$.

Equations of a Straight Line.

1. Intercept Form (Fig. 2): $\dfrac{x}{a} + \dfrac{y}{b} = 1$. ($a$, b = intercepts of the line on the axes.)

2. Slope Form (Fig. 3): $y = mx + b$. ($m = \tan u$ = slope; b = intercept on the y-axis; see also Fig. 2, p. 174.)

3. Normal Form (Fig. 4): $x \cos v + y \sin v = p$. (p = perpendicular from origin to line; v = angle p makes with the x-axis.)

4. Parallel-intercept Form (Fig. 5): $\dfrac{y - b}{c - b} = \dfrac{x}{k}$. ($b$, c = intercepts on two parallels at distance k apart.)

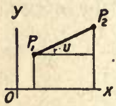


FIG. 1.

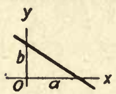


FIG. 2.

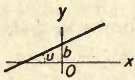


FIG. 3.

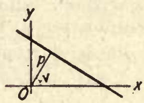


FIG. 4.

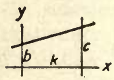


FIG. 5.

5. General Form: $Ax + By + C = 0$. [Here $a = -C/A$, $b = -C/B$, $m = -A/B$, $\cos v = A/R$, $\sin v = B/R$, $p = -C/R$, where $R = \pm\sqrt{A^2 + B^2}$ (sign to be so chosen that p is positive).]

6. Line Through (x_1, y_1) with Slope m: $y - y_1 = m(x - x_1)$.

7. Line Through (x_1, y_1) and (x_2, y_2): $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

8. Line Parallel to x-axis: $x = a$; to y-axis: $y = b$.

Angles and Distances.

If u = angle between two lines whose slopes are m_1, m_2, then

$$\tan u = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

If parallel, $m_1 = m_2$.
If perpendicular, $m_1 m_2 = -1$.

If u = angle between the lines $Ax + By + C = 0$ and $A'x + B'y + C' = 0$, then

$$\cos u = \frac{AA' + BB'}{\pm\sqrt{(A^2 + B^2)(A'^2 + B'^2)}}.$$

If parallel, $A/A' = B/B'$.
If perpendicular, $AA' + BB' = 0$.

The equations of the bisectors of the angles between the two lines just mentioned are

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} \mp \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = 0.$$

The equation of a line through (x_1, y_1) and meeting a given line $y = mx + b$ at an angle u, is

$$y - y_1 = \frac{m + \tan u}{1 - m \tan u}(x - x_1).$$

The distance from (x_0, y_0) to the line $Ax + By + C = 0$ is

$$D = \left|\frac{Ax_0 + By_0 + C}{\sqrt{A^2 + B^2}}\right|$$

where the vertical bars mean "the absolute value of."

The distance from (x_0, y_0) to a line which passes through (x_1, y_1) and makes an angle u with the x-axis, is

$$D = (x_0 - x_1)\sin u - (y_0 - y_1)\cos u.$$

Polar Co-ordinates (Fig. 6). Let (x, y) be the rectangular and (r, θ) the polar co-ordinates of a given point P. Then $x = r\cos\theta$; $y = r\sin\theta$; $x^2 + y^2 = r^2$.

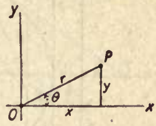


FIG. 6.

Transformation of Co-ordinates. If origin is moved to point (x_0, y_0), the new axes being parallel to the old, $x = x_0 + x'$, $y = y_0 + y'$.

If axes are turned through the angle u, without change of origin,

$$x = x'\cos u - y'\sin u, \quad y = x'\sin u + y'\cos u.$$

THE CIRCLE

(See also pp. 99, 103–105, 106)

Equation of Circle with center (a,b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2.$$

If center is at the origin, the equation becomes $x^2 + y^2 = r^2$. If circle goes through the origin and center is on the x-axis at point $(r, 0)$, equation becomes $x^2 + y^2 = 2rx$. The **general equation** of a circle is

$x^2 + y^2 + Dx + Ey + F = 0$; it has center at $(-D/2, -E/2)$, and radius $= \sqrt{(D/2)^2 + (E/2)^2 - F}$ (which may be real, null, or imaginary).

The **equation of the radical axis** of two circles, $x^2 + y^2 + Dx + Ey + F = 0$ and $x^2 + y^2 + D'x + E'y + F' = 0$, is $(D - D')x + (E - E')y + (F - F') = 0$. The tangents drawn to two circles from any point of their radical axis are of equal length. If the circles intersect, the radical axis passes through their points of intersection (see p. 100).

The **equation of the tangent** to $x^2 + y^2 = r^2$ at (x_1, y_1) is $x_1x + y_1y = r^2$. The tangent to $x^2 + y^2 + Dx + Ey + F = 0$ at (x_1, y_1) is $x_1x + y_1y + \frac{1}{2}D(x + x_1) + \frac{1}{2}E(y + y_1) + F = 0$. The line $y = mx + b$ will be tangent to the circle $x^2 + y^2 = r^2$ if $b = a\sqrt{1 + m^2}$.

Equations of Circle in Parametric Form. It is sometimes convenient to express the co-ordinates x and y of the moving point P (Fig. 7) in terms of an auxiliary variable, called a **parameter**. Thus, if the parameter be taken as the angle u which the radius OP makes with the x-axis, then the equations of the circle in parametric form will be $x = a\cos u$; $y = a\sin u$. For every value of the parameter u, there corresponds a point (x, y) on the circle. The ordinary equation $x^2 + y^2 = a^2$ can be obtained from the parametric equations by eliminating u.

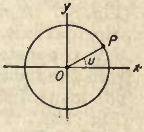


FIG. 7.

THE PARABOLA

The **parabola** (see also p. 107) is the locus of a point which moves so that its distance from a fixed line (called the **directrix**) is always equal to its distance from a fixed point F (called the **focus**). See Fig. 8. The point half-way from focus to directrix is the **vertex**, O, The line through the focus, perpendicular to the directrix, is the **principal axis.** The breadth of the curve at the focus is called the **latus rectum, or parameter,** $= 2p$, where p is the distance from focus to directrix. (Compare also Fig. 3, p. 174.)

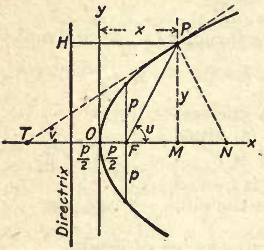


FIG. 8.

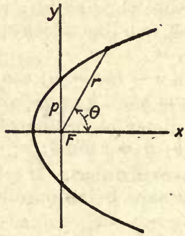


FIG. 9.

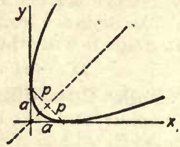


FIG. 10.

Any section of a right circular cone made by a plane parallel to a tangent plane of the cone will be a parabola.

Equation of Parabola, origin at vertex (Fig. 8): $y^2 = 2px$.

Polar Equation of Parabola, referred to F as origin and Fx as axis (Fig. 9): $r = p/(1 - \cos\theta)$.

Equation Referred to the Tangents at the ends of the latus rectum as axes (Fig. 10): $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$, where $a = p\sqrt{2}$.

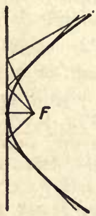


FIG. 11.

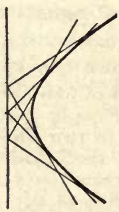


FIG. 12.

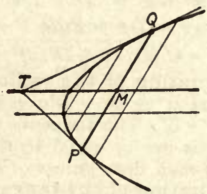


FIG. 13.

Equation of Tangent to $y^2 = 2px$ at (x_1,y_1): $y_1y = p(x + x_1)$. The line $y = mx + b$ will be tangent to $y^2 = 2px$ if $b = p/(2m)$.

The **tangent** PT at any point P bisects the angle between PF and PH (Fig. 8). A ray of light from F, reflected at P, will move off parallel to the principal axis. The **subtangent,** TM, is bisected at O. The **subnormal,** MN, is constant, and equal to p. The locus of the foot of the perpendicular from the focus on a moving tangent is the tangent at the vertex (Fig. 11). The locus of the point of intersection of perpendicular tangents is the directrix (Fig. 12). The locus of the mid-points of a set of parallel chords whose slope is m is a straight line parallel to the principal axis at a distance p/m,

and is called a **diameter** (Fig. 13). If M is the mid-point of a chord PQ, and if T is the point of intersection of the tangents at P and Q, then TM is parallel to the principal axis, and is bisected by the curve (Fig. 13).

To Construct a Tangent to a Given Parabola. (1) At a given point of contact, P (Fig. 14): Find T so that $OT = OM$, or $FT = FP$. Then TP is the tangent at P. Or, make $MN = p = 2(OF)$; then PN is the normal at P.

(2) From a given external point, Q (Fig. 15): With Q as center and radius QF draw circle cutting the directrix in H; draw HP parallel to principal axis; then P is required point of contact. As check, note that QP is the perpendicular bisector of FH.

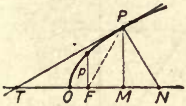


FIG. 14.

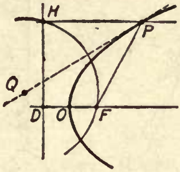


FIG. 15.

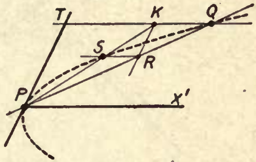


FIG. 16.

To Construct a Parabola. 1. GIVEN ANY TWO POINTS, P AND Q, THE TANGENT PT AT ONE OF THEM, AND THE DIRECTION OF THE PRINCIPAL AXIS OX. In Fig. 16, let K be a variable point on a line through Q parallel to OX. Draw KR parallel to PT (meeting PQ in R), and draw RS parallel to OX (meeting PK in S); then S is a point of the curve. NOTE. A line through P parallel to the principal axis bisects all chords parallel to the tangent PT.

2. GIVEN THE VERTEX O AND FOCUS F. (*a*) In Fig. 17 draw Oy perpendicular to OF, and slide the vertex of a right angle along Oy so that one side always passes through F; then the other side will always be a tangent to the parabola.

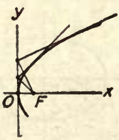


FIG. 17.

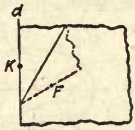


FIG. 18.

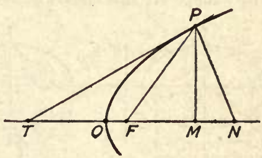


FIG. 19.

(*b*) Take a piece of paper (Fig. 18) with a straight edge, d, and mark a point F near the edge. Let K be a variable point of the edge, and fold the paper so that K coincides with F. The crease will be a tangent to the parabola which has focus F and directrix d.

(*c*) In Fig. 19, let M be a variable point of the principal axis, and lay off $MN = 2(OF) = p$. With F as center and radius FN draw a circle, cutting the perpendicular at M in P. Then P is a point of the curve, and PT and PN are the tangent and normal at P.

3. GIVEN TWO TANGENTS AND THEIR POINTS OF CONTACT, P AND Q (Fig. 20). Divide TP and QT into any number of equal parts (here 4). Then the lines 11, 22, 33, . . . will be tangents to the parabola. This method is especially advantageous in drawing rather flat arcs.

The Radius of Curvature of $y^2 = 2px$ at a point $P = (x,y)$ is $R = (p + 2x)^{3/2}/\sqrt{p}$, or $R = p/\sin^3 v$, where v = the angle which the tangent at P makes with PF (Fig. 21). At the vertex, $R = p$. To construct the radius of curvature at any point P, lay off $PR = 2(PF)$ parallel to the principal axis, and draw RC perpendicular to the axis, meeting the normal, PN, in C. Then C is the center of curvature for the point P, and a circle about C with radius CP will coincide closely with the parabola in the neighborhood of P.

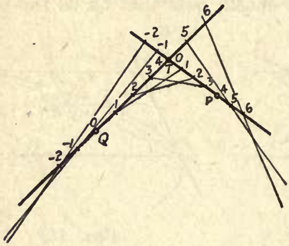


FIG. 20.

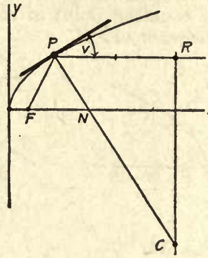


FIG. 21.

THE ELLIPSE

The **ellipse** (see also p. 107) has two **foci,** F and F' (Fig. 22), and two **directrices,** DH and $D'H'$. If P is any point of the curve, $PF + PF'$ is constant, $= 2a$; and PF/PH (or PF'/PH') is also constant, $= e$, where e is the **eccentricity** ($e < 1$). Either of these properties may be taken as the definition of the curve. The relations between e and the semi-axes a and b are as shown in Fig. 23. Thus, $b^2 = a^2(1 - e^2)$, $ae = \sqrt{a^2 - b^2}$, $e^2 = 1 - (b/a)^2$. The **semi-latus rectum** $= p = a(1 - e^2) = b^2/a$. Note that b is always less than a, except in the special case of the circle, in which $b = a$ and $e = 0$.

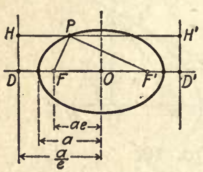


FIG. 22.

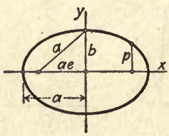


FIG. 23.

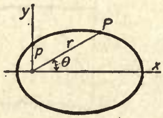


FIG. 24.

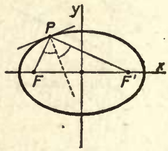


FIG. 25.

Any section of a right circular cone made by a plane which cuts all the elements of one nappe of the cone will be an ellipse; if the plane is perpendicular to the axis of the cone, the ellipse becomes a circle.

Equation of Ellipse, center as origin:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ or } y = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

If $P = (x, y)$ is any point of the curve, $PF = a + ex$, $PF' = a - ex$.

Equations of the Ellipse in Parametric Form: $x = a \cos u$, $y = b \sin u$, where u is the eccentric angle of the point $P = (x,y)$. See Fig. 28.

Polar Equation, focus as origin, axes as in Fig. 24: $r = p/(1 - e\cos\theta)$.

Equation of the Tangent at (x_1,y_1): $b^2x_1x + a^2y_1y = a^2b^2$.

The line $y = mx + k$ will be a tangent if $k = \pm\sqrt{a^2m^2 + b^2}$. The normal at any point P bisects the angle between PF and PF' (Fig. 25). The locus of the foot of the perpendicular from the focus on a moving tangent is the circle on the major axis as diameter (Fig. 26). The locus of the point of intersection of perpendicular tangents is a circle with radius $\sqrt{a^2 + b^2}$ (Fig. 27).

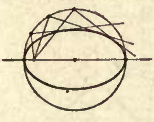


FIG. 26.

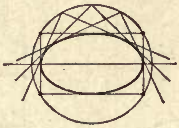


FIG. 27.

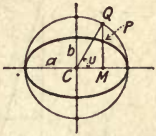


FIG. 28.

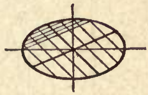


FIG. 29.

Ellipse as a Flattened Circle. Eccentric Angle. If the ordinates in a circle are diminished in a constant ratio, the resulting points will lie on an ellipse (Fig. 28). If Q traces the circle with uniform velocity, the corresponding point P will trace the ellipse, with varying velocity. The angle u in the figure is called the eccentric angle of the point P.

Conjugate Diameters are lines through the center, each of which bisects all the chords parallel to the other (Fig. 29). If m_1 and m_2 are the slopes, then $m_1m_2 = -b^2/a^2$. One pair of conjugate diameters are the diagonals of the rectangle circumscribing the ellipse. The eccentric angles of the ends of two conjugate diameters differ by 90 deg. Thus (Fig. 30), if CQ and CQ' are perpendicular radii in the circle, CP and CP' will be conjugate semi-diameters in the ellipse. A parallelogram formed by tangents drawn parallel to a pair of conjugate diameters has a constant area, $= 4ab$ (Fig. 31). Also, if a',b' are conjugate semi-diameters, and w the angle between them, then $a'^2 + b'^2 = a^2 + b^2$ and $a'b' = ab/\sin w$.

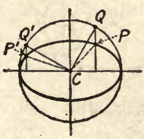


FIG. 30.

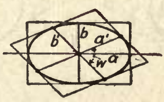


FIG. 31.

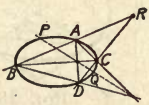


FIG. 32.

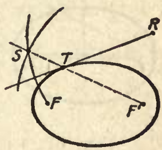


FIG. 33.

To Construct a Tangent to a Given Ellipse. (1) AT A GIVEN POINT OF CONTACT, P. Bisect the angle between the focal radii PF and PF' (Fig. 25).

(2) FROM A GIVEN EXTERNAL POINT, R. (*a*) Through R draw any two lines cutting the ellipse, one in A and B, the other in C and D (Fig. 32). Through the point of intersection of AD and BC and the point of intersection of AC and BD, draw a line cutting the ellipse in P and Q. Then P and Q are the required points of contact. (*b*) With R as a center and radius RF, draw an arc; with F' as center and radius $2a$ draw an arc, intersecting the first in S; and let SF' meet the curve in T. Then T is the point of contact (Fig. 33).

To Construct an Ellipse, Given a and b. (1) In Fig. 34, with O as center, draw circles with radii a and b (and also a third circle with radius $a + b$). Let a variable ray through O cut these circles in J, K (and S); through J and K draw parallels to the axes, meeting in P. Then P is a point of the ellipse (and SP is the normal at P).

(2) In Fig. 35, let P divide a line AB so that $PA = a$ and $PB = b$. Then if A and B slide on the axes, P will describe an ellipse.

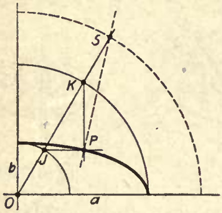


FIG. 34.

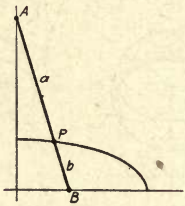


FIG. 35.

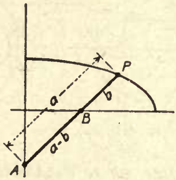


FIG. 36.

(3) In Fig. 36, let PBA be a straight line such that $PA = a$ and $PB = b$. Then if A and B slide on the axes, P will trace an ellipse. (Use a strip of paper, with the points P, B, and A marked on it.)

(4) Find the foci, F and F', by striking an arc of radius a with center at B (Fig. 37). Drive pins at F, F', and B, and adjust a loop of thread around them. Then remove the pin at B, and replace it by a pencil point; by moving the pencil so as to keep the string taut, the complete ellipse can be drawn at one sweep. Or, use a mechanical ellipsograph.

(5) and (6). Apply methods (1) and (2) of the following paragraph to the special case in which OP and OQ are perpendicular semi-axes.

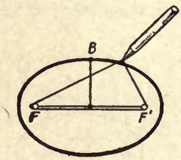


FIG. 37.

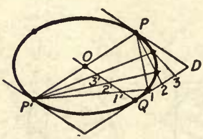


FIG. 38.

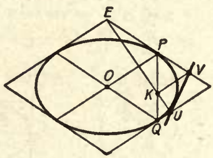


FIG. 39.

To Construct an Ellipse, Given a Pair of Conjugate Semi-diameters, OP and OQ. (1) Complete the parallelogram, as in Fig. 38. Divide QD and QO into n equal parts, 1, 2, 3, . . . and 1′, 2′, 3′, . . . Connect P with 1, 2, 3, . . . and P' with 1′, 2′, 3′ . . . . The points of intersection of corresponding lines will be points of the ellipse.

(2) Take any point K on PQ (Fig. 39). Draw EKU, and draw KV parallel to OP. Then UV will be a tangent. By varying K along PQ as many tangents may be drawn as desired, thus "enveloping" the ellipse.

(3) Through P (Fig. 40), draw a perpendicular PT to OQ, and lay off $PR = PS = OQ$. Then if the line RPT is made to slide with one end on OR and the other on OQ, P will trace the ellipse. Further, the bisectors of the angle ROS show the directions of the principal axes, and $OR + OS = 2a$ and

$OR - OS = 2b$. Also, if a line through P perpendicular to RS (and therefore tangent to the ellipse at P) meets the minor axis in M, a circle with M as center and MR or MS as radius will cut the major axis in the two foci.

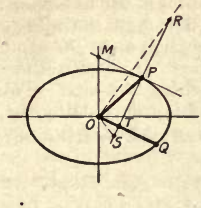


FIG. 40.

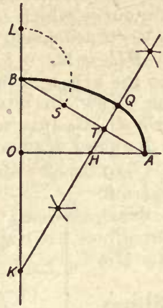


FIG. 41.

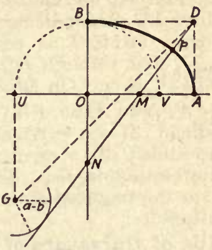


FIG. 42.

To Construct an Ellipse Approximately by Circular Arcs. [Methods (1) and (2) employ two radii, (3) and (4) employ three radii.] (1) In Fig. 41, lay off $OL = OA$ and $BS = BL = a - b$. Bisect SA in T, and draw THK perpendicular to BA. Then H is one center, with radius HA, and K is the other center, with radius KB. The junction point Q of the two arcs will fall a little outside the true ellipse.

(2) In Fig. 42, lay off $OU = OV = OB = b$. Draw UG perpendicular to the axis and DG at 45°. With G as center draw an auxiliary arc with radius

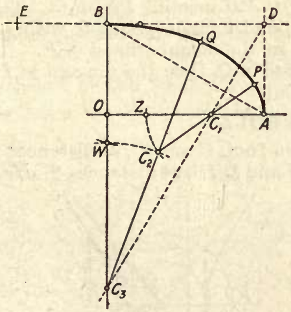


FIG. 43.

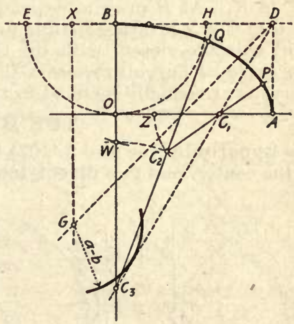


FIG. 44.

$= AV = a - b$, and through D draw DMN just touching this arc. Then M is one center (with radius MA) and N is the other center (with radius NB). The junction point P of the two arcs will be a true point of the ellipse. [E. V. Huntington.]

(3) Through D (Fig. 43) draw DC_1C_3 perpendicular to AB. Call $C_1A = r_1$ and $C_3B = r_3$. Lay off $BE = BO$ ($= b$), and on ED as diameter draw a semicircle cutting the minor axis in W; then $BW = \sqrt{ab} = r_2$. Lay off $AZ =$

BW. From C_1 with radius $C_1Z(=r_2-r_1)$, and from C_3 with radius C_3W $(=r_3-r_2)$, draw arcs intersecting in C_2. Draw C_3C_2 extended and C_2C_1 extended. Then draw in the three arcs, with centers at C_1, C_2, C_3 and radii r_1, r_2, r_3. NOTE. Since r_1 and r_3 are the radii of curvature of the ellipse at A and B, this construction gives a curve which is a little too sharp at A and a little too flat at B. A more accurate construction is the following:

(4) In Fig. 44, lay off $BE = BH = BO = b$. Through the mid-point X of BE draw XG perpendicular to the axis, and through D draw DG at an angle of 45 deg. From G as center draw an auxiliary arc with radius $= DH$ $(= a - b)$, and through D draw DC_1C_3 just touching this arc. Take C_1A as r_1 and C_3B as r_3. On DE as diameter draw a semi-circle cutting the minor axis in W, and take $BW(=\sqrt{ab})$ as r_2. Lay off $AZ = BW$. From C_1 with radius $C_1Z(=r_2-r_1)$, and from C_3 with radius $C_3W(=r_3-r_2)$, draw arcs intersecting in C_2. Then C_1, C_2, C_3 are the required centers. [E. V. Huntington.]

Radius of Curvature of Ellipse at Any Point $P = (x, y)$ is $R = a^2b^2(x^2/a^4 + y^2/b^4)^{3/2} = p/\sin^3 v$, where v is the angle which the tangent at P makes with PF or PF'. At end of major axis, $R = b^2/a = MA$; at end of minor axis, $R = a^2/b = NB$ (see Fig. 45). To construct the radius of curvature at any other point P (Fig. 46), draw the normal at P (by bisecting the angle between PF and PF') and let it meet the major axis in N. At N draw a perpendicular to PN meeting PF in H. At H draw a perpendicular to PH meeting PN in C. Then C is the center of curvature for the point P, and a circle about C with radius CP will coincide closely with the ellipse in the neighborhood of P. [Note. If the circle of curvature meets the ellipse in Q, then the tangent at P and the line PQ are equally inclined to the axis.]

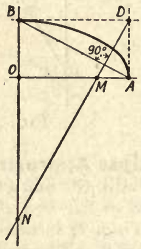


FIG. 45.

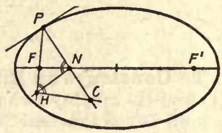


FIG. 46.

THE HYPERBOLA

The **hyperbola** (see also p. 107) has two **foci,** F and F', at distances $\pm\, ae$ from the center, and two **directrices,** DH and $D'H'$, at distances $\pm\, a/e$ from

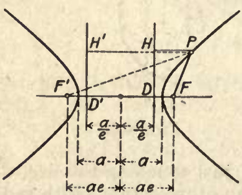


FIG. 47.

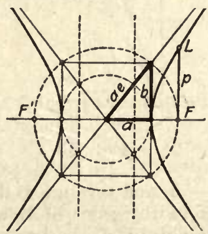


FIG. 48.

the center (Fig. 47). If P is any point of the curve, $|PF - PF'|$ is constant, $= 2a$; and PF/PH (or PF'/PH') is also constant, $= e$ (called the **eccentricity**), where $e > 1$. Either of these properties may be taken as the definition of the

curve. The curve has two branches which approach more and more nearly two straight lines called the **asymptotes.** Each asymptote makes with the principal axis an angle whose tangent is b/a. The relations between e, a, and b are shown in Fig. 48: $b^2 = a^2(e^2 - 1)$, $ae = \sqrt{a^2 + b^2}$, $e^2 = 1 + (b/a)^2$. The semi-latus rectum, or ordinate at the focus, is $p = a(e^2 - 1) = b^2/a$.

Any section of a right circular cone made by a plane which cuts both nappes of the cone will be a hyperbola. (Compare also Fig. 3, p. 174.)

Equation of the Hyperbola, center as origin:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ or } y = \pm \frac{b}{a}\sqrt{x^2 - a^2}.$$

If $P = (x,y)$ is on the right-hand branch, $PF = ex - a$, $PF' = ex + a$. If P is on the left-hand branch, $PF = -ex + a$, $PF' = -ex - a$.

Equations of Hyperbola in Parametric Form. (1) $x = a \cosh u$, $y = b \sinh u$. (For tables of hyperbolic functions, see pp. 60 and 61.) Here u may be interpreted as A/a^2, where A is the area shaded in Fig. 49.

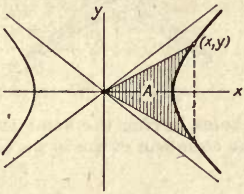


FIG. 49.

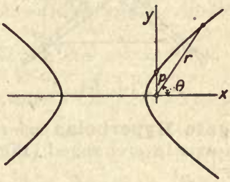


FIG. 50.

(2) $x = a \sec v$, $y = b \tan v$, where v is an auxiliary angle of no special geometric interest.

Polar Equation, referred to focus as origin, axes as in Fig. 50:

$$r = p/(1 - e \cos \theta).$$

Equation of the Tangent at (x_1,y_1): $b^2x_1x - a^2y_1y = a^2b^2$.

The line $y = mx + k$ will be a tangent if $k = \pm \sqrt{a^2m^2 - b^2}$. The tan-

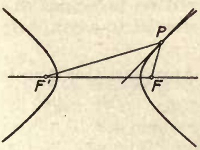


FIG. 51.

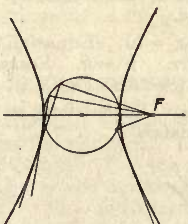


FIG. 52.

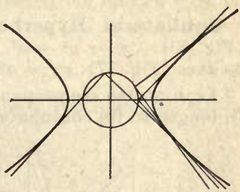


FIG. 53.

gent at any point P (Fig. 51) bisects the angle between PF and PF'. The locus of the foot of the perpendicular from the focus on a moving tangent is the circle on the principal axis as diameter (Fig. 52). The locus of the point of intersection of perpendicular tangents is a circle with radius $\sqrt{a^2 - b^2}$, which will be imaginary if $b > a$ (Fig. 53).

10

Properties of the Asymptotes. (Fig. 54.) If P is any point of the curve, the product of the perpendicular distances from P to the two asymptotes is constant, $= a^2b^2/(a^2 + b^2)$. Also, the product of the oblique distances (the distance to each asymptote being measured parallel to the other) is constant, and equal to $\frac{1}{4}(a^2 + b^2)$. If a line cuts the hyperbola and its asymptotes, the parts of the line intercepted between the curve and the asymptotes are equal. The part of a tangent intercepted between the asymptotes is bisected by the point of contact. The triangle bounded by the asymptotes and a variable tangent is of constant area, $= ab$. If a line through Q perpendicular to the principal axis meets the asymptotes in R and S (see Fig. 54), then $\overline{QR} \times \overline{QS} = b^2$. If a line through Q parallel to the principal axis meets the asymptotes in U and V, then $\overline{QU} \times \overline{QV} = a^2$.

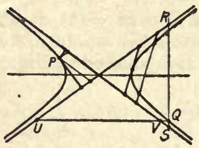


FIG. 54.

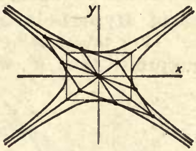


FIG. 55.

Conjugate Hyperbolas are two hyperbolas having the same asymptotes with semi-axes interchanged (Fig. 55). The equation of the hyperbola conjugate to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$.

Conjugate Diameters are lines through the center, each of which bisects all the chords parallel to the other—a chord which does not meet the given hyperbola being understood to be terminated by the conjugate hyperbola (Fig. 55). If m_1 and m_2 are the slopes, then $m_1m_2 = b^2/a^2$. Each asymptote, regarded as a diameter, is its own conjugate. If a parallelogram is formed by tangents drawn parallel to a pair of conjugate diameters, its vertices will lie on the asymptotes, and its area will be constant $= 4ab$. If a', b' are conjugate semi-diameters, and w the angle between them, then $a'^2 - b'^2 = a^2 - b^2$, and $a'b' = ab/\sin w$.

Equilateral Hyperbola ($a = b$). Equation referred to principal axes (Fig. 56): $x^2 - y^2 = a^2$. NOTE. $p = a$. Equation referred to asymptotes as axes (Fig. 57): $xy = a^2/2$. (See also Fig. 3, p. 174.)

Asymptotes are perpendicular. Eccentricity $= \sqrt{2}$. Any diameter is equal in length to its conjugate diameter.

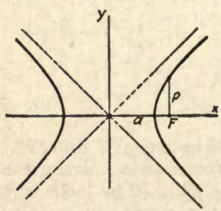


FIG. 56.

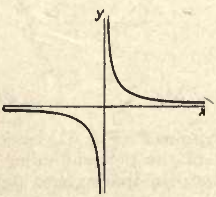


FIG. 57.

To Construct a Tangent at any given point P of a hyperbola. In Fig. 58, draw PA and PB parallel to the asymptotes, and take $OS = 2(OA)$ and $OT = 2(OB)$. Then ST is the tangent at P.

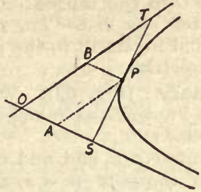


FIG. 58.

To Construct a Hyperbola, given the asymptotes and any point P. (1) In Fig. 59 let TPT' be a variable line through P, and lay off $T'P' = TP$; then P' is a point of the curve.

(2) In Fig. 60, draw PA and PB parallel to the asymptotes. Lay off $OA' = n(OA)$ and $OB' = (1/n)(OB)$, where n is any number; and through A' and B' draw parallels to the axes; these will meet in a point P' of the curve.

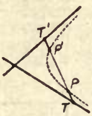


FIG. 59.

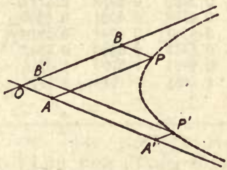


FIG. 60.

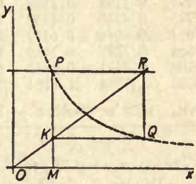


FIG. 61.

(3) (Fig. 61.) Take any point K in the ordinate PM, and draw OK meeting the line through P parallel to the x-axis in R. Draw a parallel to the x-axis through K and a parallel to the y-axis through R, meeting in Q. Then Q is a point of the curve.

THE CATENARY

The catenary is the curve in which a flexible chain or cord of uniform density will hang when supported by the two ends. Let w = weight of the chain per unit length; T = the tension at any point P; and T_h, T_v = the horizontal and vertical components of T. The horizontal component T_h is the same at all points of the curve.

The length $a = T_h/w$ is called the **parameter** of the catenary, or the distance from the lowest point O to the **directrix** DQ (Fig. 62). When a is very large, the curve is very flat. For methods of finding a in any given case, see problems 1–6 below.

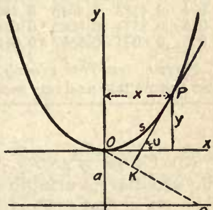


FIG. 62.

The rectangular **equation**, referred to the lowest point as origin, is $y = a\ [\cosh\ (x/a) - 1]$. (For table of hyperbolic functions, see p. 60.) In case of

very flat arcs (a large), $y = \frac{x^2}{2a} + \ldots$; $s = x + \frac{1}{6}\frac{x^3}{a^2} + \ldots$, approximately, so that in such a case the catenary closely resembles a parabola.

If the perpendicular from O to the tangent at P meets the directrix in Q, then $DQ = \text{arc}\, OP = s$ and $OQ = y + a$. The **radius of curvature** at P is $R = (y + a)^2/a$, which is equal in length to the portion of the normal intercepted between P and the directrix.

Problems on the Catenary (Fig. 62). When any two of the four quantities x, y, s, T/w are known, the remaining two, and also the parameter a, can be found, as follows:

(1) GIVEN x AND y. Compute y/x, and find from Table 1 the value of the auxiliary variable z. Then compute $a = x/z$, $s = a \sinh z$, and $T = wa \cosh z$. Or, having z, find s/x and wx/T by using Tables 3 and 2 inversely, and hence (since x is known) compute s and T/w without the use of a.

TABLE 1. GIVING z WHEN y/x IS KNOWN. THEN $a = x/z$

y/x	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0200	0.0400	0.0600	0.0800	0.0999	0.1199	0.1398	0.1597	0.1795
0.1	0.1993	0.2191	0.2389	0.2586	0.2782	0.2978	0.3173	0.3368	0.3562	0.3756
0.2	0.3948	0.4140	0.4332	0.4522	0.4712	0.4901	0.5089	0.5276	0.5463	0.5648
0.3	0.5833	0.6016	0.6199	0.6381	0.6561	0.6741	0.6919	0.7097	0.7274	0.7449
0.4	0.7623	0.7797	0.7969	0.8140	0.8311	0.8480	0.8647	0.8814	0.8980	0.9145
0.5	0.9308	0.9471	0.9632	0.9792	0.9951	1.0109	1.0266	1.0422	1.0576	1.0730
0.6	1.0883	1.1034	1.1184	1.1334	1.1482	1.1629	1.1775	1.1920	1.2064	1.2207

NOTE. $y/x = (\cosh z - 1)/z$.

(2) GIVEN x AND T/w. Compute wx/T, and find from Table 2 the value of the auxiliary variable z. Then compute $a = x/z$, $y = a(\cosh z - 1)$ and $s = a \sinh z$. Or, having z, find y/x and s/x by using Tables 1 and 3 inversely, and hence (since x is known) compute y and s without the use of a.

TABLE 2. GIVING z WHEN wx/T IS KNOWN. THEN $a = x/z$

wx/T	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0100	0.0200	0.0300	0.0400	0.0501	0.0601	0.0702	0.0803	0.0904
0.1	0.1005	0.1107	0.1209	0.1311	0.1414	0.1517	0.1621	0.1725	0.1830	0.1936
0.2	0.2042	0.2149	0.2256	0.2365	0.2474	0.2584	0.2695	0.2807	0.2920	0.3035
0.3	0.3150	0.3267	0.3385	0.3505	0.3626	0.3749	0.3874	0.4000	0.4129	0.4259
0.4	0.4392	0.4528	0.4666	0.4806	0.4950	0.5097	0.5248	0.5403	0.5562	0.5726
0.5	0.5894	0.6068	0.6249	0.6436	0.6632	0.6836	0.7051	0.7277	0.7517	0.7775
0.6	0.8053	0.8357	0.8695	0.9082	0.9541	1.0132	1.1110	.....	......	......

NOTE. $wx/T = z/\cosh z$. For every value of wx/T there are two values of z, one less than 1.200 and one greater than 1.200. Only the smaller of these values is tabulated.

(3) GIVEN x AND s. Compute s/x, and find from Table 3 the value of the auxiliary variable z. Then compute $a = x/z$, $y = a (\cosh z - 1)$, and $T = wa \cosh z$. Or, having z, find y/x and wx/T by using Tables 1 and 2 inversely, and hence (since x is known) compute y and T/w without the use of a.

TABLE 3. GIVING z WHEN s/x IS KNOWN. THEN $a = x/z$

s/x	0	1	2	3	4	5	6	7	8	9
1.000	......	0.0245	0.0346	0.0424	0.0490	0.0548	0.0600	0.0648	0.0693	0.0735
1	0.0774	0.0812	0.0848	0.0883	0.0916	0.0948	0.0980	0.1010	0.1039	0.1067
2	0.1095	0.1122	0.1149	0.1174	0.1200	0.1224	0.1249	0.1272	0.1296	0.1319
3	0.1341	0.1363	0.1385	0.1407	0.1428	0.1448	0.1469	0.1489	0.1509	0.1529
4	0.1548	0.1567	0.1586	0.1605	0.1623	0.1642	0.1660	0.1678	0.1696	0.1713
1.005	0.1731	0.1748	0.1765	0.1782	0.1799	0.1815	0.1831	0.1848	0.1864	0.1880
6	0.1896	0.1911	0.1927	0.1942	0.1958	0.1973	0.1988	0.2003	0.2018	0.2033
7	0.2047	0.2062	0.2076	0.2091	0.2105	0.2119	0.2133	0.2147	0.2161	0.2175
8	0.2188	0.2202	0.2215	0.2229	0.2242	0.2255	0.2269	0.2282	0.2295	0.2308
9	0.2321	0.2334	0.2346	0.2359	0.2372	0.2384	0.2397	0.2409	0.2421	0.2434
1.01	0.2446	0.2565	0.2678	0.2787	0.2892	0.2993	0.3091	0.3186	0.3278	.3367
2	0.3454	0.3539	0.3621	0.3702	0.3781	0.3859	0.3934	0.4009	0.4082	0.4153
3	0.4224	0.4293	0.4361	0.4428	0.4494	0.4559	0.4623	0.4686	0.4748	0.4809
4	0.4870	0.4930	0.4989	0.5047	0.5105	0.5162	0.5218	0.5274	0.5329	0.5383
1.05	0.5437	0.5490	0.5543	0.5595	0.5647	0.5698	0.5749	0.5799	0.5849	0.5898
6	0.5947	0.5996	0.6044	0.6091	0.6139	0.6186	0.6232	0.6278	0.6324	0.6369
7	0.6414	0.6459	0.6504	0.6548	0.6591	0.6635	0.6678	0.6721	0.6763	0.6806
8	0.6848	0.6889	0.6931	0.6972	0.7013	0.7053	0.7094	0.7134	0.7174	0.7213
9	0.7253	0.7292	0.7331	0.7369	0.7408	0.7446	0.7484	0.7522	0.7559	0.7597
1.10	0.7634	......	......	......	......	......	NOTE: $s/x = \sinh z/z$			

(4) GIVEN y AND s. Then $\dfrac{T}{w} = \dfrac{s^2}{2y} + \dfrac{y}{2}$, $x = \left(\dfrac{s^2}{y} - y\right)\tanh^{-1}\left(\dfrac{y}{s}\right)$,

$a = \dfrac{s^2}{2y} - \dfrac{y}{2}$. Or, if y/s is small, $x = s\left[1 - \dfrac{2}{3}\left(\dfrac{y}{s}\right)^2 - \dfrac{2}{15}\left(\dfrac{y}{s}\right)^4 - \cdots\right]$.

(5) GIVEN y AND T/w. Then $a = \dfrac{T}{w} - y$, $x = \left(\dfrac{T}{w} - y\right)\cosh^{-1}\dfrac{T/w}{(T/w) - y}$,

$s = \sqrt{2y(T/w) - y^2}$. Or, if $y/(T/w)$ is small,

$$x = \sqrt{\frac{2yT}{w}}\left[1 - \frac{7}{12}\frac{wy}{T} - \cdots\right], \quad \frac{s - x}{s} = \frac{1}{3}\frac{wy}{T}, \text{ approximately,}$$

$$s = \sqrt{\frac{2yT}{w}}\left[1 - \frac{1}{4}\frac{wy}{T} - \frac{1}{32}\left(\frac{wy}{T}\right)^2 - \frac{1}{128}\left(\frac{wy}{T}\right)^3 - \cdots\right].$$

(6) GIVEN s AND T/w. Then $x = \dfrac{T}{w}\sqrt{1 - \left(\dfrac{ws}{T}\right)^2}\tanh^{-1}\left(\dfrac{ws}{T}\right)$,

$y = \dfrac{T}{w} - \dfrac{T}{w}\sqrt{1 - \left(\dfrac{ws}{T}\right)^2}$, $a = \dfrac{T}{w}\sqrt{1 - \left(\dfrac{ws}{T}\right)^2}$. Or, if ws/T is small,

$$x = s\left[1 - \frac{1}{6}\left(\frac{ws}{T}\right)^2 - \frac{11}{120}\left(\frac{ws}{T}\right)^4 - \cdots\right], \quad y = s\left[\frac{1}{2}\left(\frac{ws}{T}\right) + \frac{1}{8}\left(\frac{ws}{T}\right)^3 + \cdots\right]$$

$$a = \frac{T}{w}\left[1 - \frac{1}{2}\left(\frac{ws}{T}\right)^2 - \frac{1}{8}\left(\frac{ws}{T}\right)^4 - \cdots\right].$$

Given the Length 2L of a Chain Supported at Two Points A and B not in the Same Level, to find a. (See Fig. 63; b and c are supposed known.) Let $(\sqrt{L^2 - b^2})/c = s/x$; enter Table 3 with this value of s/x, and find the corresponding value of the auxiliary variable z. Then $a = c/z$.

NOTE. The co-ordinates of the mid-point M of AB (see Fig. 63) are $x_0 = a \tanh^{-1}(b/L)$, $y_0 = (L/\tanh z) - a$, so that the position of the lowest point is determined.

Correction for Sag in Chaining Uphill (Fig. 64). Let l = length of tape (corrected for stretch and temperature), w = weight per unit length of tape, A = angle between the chord AB and the horizontal.

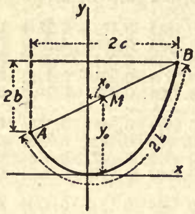


FIG. 63.

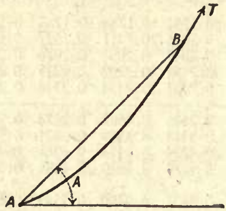


FIG. 64.

If the tension P at the upper end is known, compute wl/P and find k from Table 4. If the tension Q at the lower end is known, compute wl/Q and find k from Table 5. In either case, chord $AB = l - kl$.

TABLE 4. GIVING k

$\frac{wl}{P}$	$A = 0°$	10°	20°	30°	40°	50°	60°	70°	80°
.01	.00000	000	000	000	000	000	000	000	000
.02	002	002	001	001	001	001	000	000	000
.03	004	004	003	003	002	002	001	000	000
.04	007	006	006	005	004	003	002	001	000
.05	011	010	009	008	006	004	003	001	000
.06	.00015	015	013	012	009	006	004	002	000
.07	020	020	018	016	012	009	005	003	001
.08	027	026	024	021	016	012	007	003	001
.09	034	033	031	026	021	015	009	004	001
.10	042	041	038	033	026	019	011	005	001
.11	.00051	050	046	040	032	023	014	007	002
.12	060	060	055	048	038	027	017	008	002
.13	070	070	065	057	045	032	020	009	002
.14	082	081	076	066	053	038	023	011	003
.15	094	094	087	076	061	044	027	013	003
.16	.00107	107	100	087	070	050	031	015	004
.17	121	121	113	099	079	057	035	017	004
.18	136	136	128	112	090	065	040	019	005
.19	151	152	143	125	101	073	045	021	006
.20	168	168	159	140	113	082	050	024	006

TABLE 5. GIVING k

$\frac{wl}{Q}$	$A = 0°$	10°	20°	30°	40°	50°	60°	70°	80°
.01	.00000	000	000	000	000	000	000	000	000
.02	002	002	001	001	001	001	000	000	000
.03	004	004	003	003	002	001	001	000	000
.04	007	006	006	005	004	003	002	001	000
.05	011	010	009	008	006	004	002	001	000
.06	.00015	014	013	011	008	006	004	002	000
.07	020	020	018	015	011	008	005	002	001
.08	027	026	023	019	015	011	006	003	001
.09	034	032	029	024	019	013	008	004	001
.10	042	040	036	030	023	016	010	004	001
.11	.00051	048	043	036	028	019	011	005	001
.12	060	057	051	043	033	023	014	006	002
.13	070	067	060	050	038	026	016	007	002
.14	082	078	069	057	044	030	018	008	002
.15	094	089	079	066	050	035	021	010	002
.16	.00107	101	090	074	057	039	022	011	003
.17	121	114	101	084	064	044	026	012	003
.18	136	128	113	092	071	049	029	013	003
.19	151	142	125	103	079	054	032	015	004
.20	168	157	138	114	087	060	035	016	004

NOTE. $k = 1 - \{[1 - \sqrt{1 - 2m \sin u + m^2}]/[m \sin A]\}$, where $m = wl/P$ and u is given by

$[1 - \sqrt{1 - 2m \sin u + m^2}] \sec u = [\sinh^{-1}(\tan u) - \sinh^{-1}(\tan u - m \sec u)] \tan A.$

Also, $Q = P - wl(1 - k) \sin A$, where k is the value in Table 4 corresponding to the given values of P and A.

Correction for Stretch in Chaining Uphill. Let L = unstretched length of tape at working temperature, w = weight per unit length of tape, A = angle

between chord AB and the horizontal, F = area of cross-section, E = Young's modulus of elasticity (for steel, E = 29,000,000 lb. per sq. in.), l = stretched length (along curve).

If the tension P at the upper end is known, compute wL/P and find m from Table 6. Then $l = L + (LP/FE)(1 \mp m)$.

If the tension Q at the lower end is known, compute wL/Q and find n from Table 7. Then $l = L + (LQ/FE)(1 + n)$.

TABLE 6. GIVING m

$\frac{wL}{P}$	A = 0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
.00	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
.10	.001	.010	.018	.026	.033	.039	.044	.047	.049	.050
.20	.003	.021	.038	.053	.067	.078	.088	.094	.099	.100

TABLE 7. GIVING n

$\frac{wL}{Q}$	A = 10°	20°	30°	40°	50°	60°	70°	80°	90°
.00	.000	.000	.000	.000	.000	.000	.000	.000	000
.10	.008	.016	.024	.032	.038	.043	.047	.049	.050
.20	.014	.031	.047	.062	.075	.086	.094	.099	.100

OTHER USEFUL CURVES

The Cycloid is traced by a point on the circumference of a circle which rolls without slipping along a straight line. **Equations** of cycloid, in parametric form (axes as in Fig. 65): $x = a\,(\text{rad}\ u - \sin u)$, $y = a(1 - \cos u)$, where a is

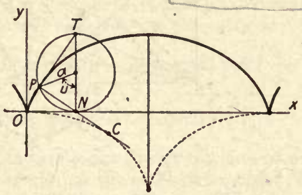


FIG. 65.

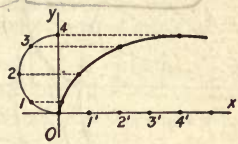


FIG. 66.

the radius of the rolling circle, and rad u is the radian measure of the angle u through which it has rolled. The tangent and normal at any point pass through the highest and lowest points of the corresponding position of the generating circle. The **radius of curvature** at any point P is PC = $4a \sin(u/2) = 2\sqrt{2ay}$ = twice the length of the normal, PN. The **evolute**, or locus of centers of curvature, is an equal cycloid. **To construct** a cycloid (Fig. 66), divide the semi-circumference of the generating circle into n equal parts (here 4) and lay off these arcs along the base (from O to $4'$). Describe arcs with centers at $1'$, $2'$, . . . and radii equal to the chords $O1$, $O2$, . . ., and sketch the cycloid as a curve tangent to all of these arcs. Or, on horizontal lines through 1, 2, . . . lay off distances equal to $O1'$, $O2'$, etc.; the points thus reached will lie on the cycloid.

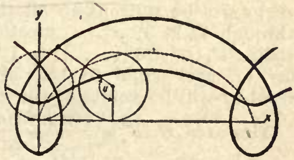
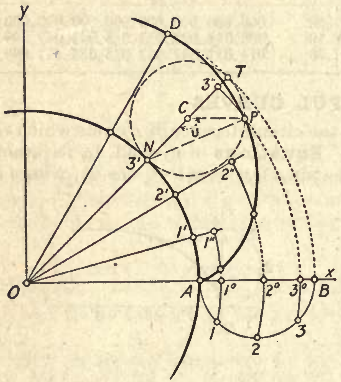


FIG. 67.

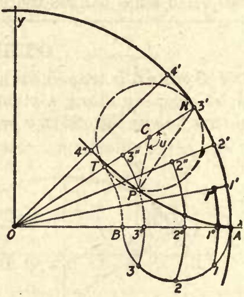
The area of one arch = $3\pi a^2$, length of arc of one arch = $8a$. Area bounded by the ordinate of the point P corresponding to any value of u is $a^2\,(\tfrac{3}{2}\ \text{rad}\ u - 2 \sin u + \tfrac{1}{4} \sin 2u) = \tfrac{3}{2}\,ax - \tfrac{1}{2}\,y\sqrt{(2a - y)y}$. Length of arc $OP = 4a\,(1 - \cos \tfrac{1}{2} u) = 4a - 2\sqrt{2a(2a - y)}$.

The Trochoid is a more general curve, traced by any point on a radius of the rolling circle, at distance b from the center (Fig. 67). It is a prolate trochoid if $b < a$, and a curtate or looped trochoid if $b > a$. The equations in either case are $x = a \text{ rad } u - b \sin u$, $y = a - b \cos u$.

The Epicycloid (or Hypocycloid) is a curve generated by a point on the circumference of a circle of radius a which rolls without slipping on the outside (or inside) of a fixed circle of radius c. For the **equations,** put $b = a$ in the equations of the epi- (or hypo-) trochoid, below. The normal at any point P passes through the point of contact N of the corresponding position of the rolling circle. To construct the curve (Figs. 68 and 69),



Epicycloid.
FIG. 68.



Hypocycloid.
FIG. 69.

divide the semi-circumference of the rolling circle into n equal parts, by points 1, 2, 3 . . . ., and lay off these arcs ($A1$, $A2$, $A3$) along the circumference of the base circle, as $A1'$, $A2'$, $A3'$, ... Describe circles with centers at $1'$, $2'$, $3'$, . . . and radii equal to the chords $A1$, $A2$, $A3$, . . .; then the required curve will be tangent to all these circles. Or, with O as center, draw arcs through 1, 2, 3, ..., meeting the radius OA in 1^0, 2^0, 3^0, . . ., and the radii $O1'$, $O2'$, $O3'$, . . . in $1''$, $2''$, $3''$, . . . ; then from $1''$, $2''$, $3''$, . . . lay off arcs equal to 1^01, 2^02, 3^03, . . . respectively; the points thus reached will be points of the curve.

The area $OAP = \dfrac{a(c \pm a)(c \pm 2a)}{2c}(\text{rad } u - \sin u)$, where the upper sign applies to the epicycloid, the lower to the hypocycloid, and rad $u =$ the radian measure of the angle u shown in Figs. 68 and 69. Arc $AP = (4\,a/c)(c \pm a)(1 - \cos \tfrac{1}{2} u)$; arc $AD = (4a/c)(c \pm a)$. [In Fig. 69, $D = 4''$.]

Radius of curvature at any point P is $R = \dfrac{4a(c \pm a)}{c \pm 2a} \sin \tfrac{1}{2}u$; at A, $R = 0$; at D, $R = \dfrac{4a(c \pm a)}{c \pm 2a}$.

Special Cases. If $a = \tfrac{1}{2}c$, the hypocycloid becomes a straight line, diameter of the fixed circle (Fig. 70). In this case the hypotrochoid traced by any

point rigidly connected with the rolling circle (not necessarily on the circumference) will be an ellipse. If $a = \frac{1}{4}c$, the curve generated will be the four-cusped hypocycloid, or **astroid,** (Fig. 71), whose equation is $x^{2/3} + y^{2/3} = c^{2/3}$. If $a = c$, the epicycloid is the **cardioid,** whose equation in polar coordinates (axes as in Fig. 72) is $r = 2c(1 + \cos\theta)$. Length of cardioid $= 16c$.

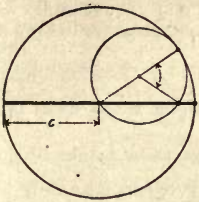
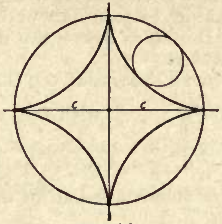
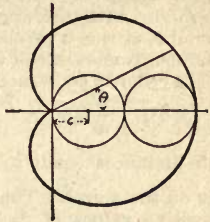


FIG. 70.



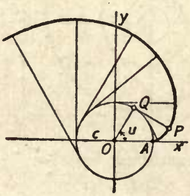
Astroid.
FIG. 71.



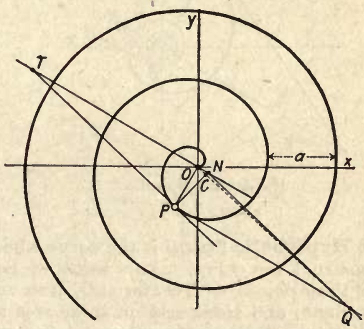
Cardioid.
FIG. 72.

The Epitrochoid (or Hypotrochoid) is a curve traced by any point rigidly attached to a circle of radius a, at distance b from the center, when this circle rolls without slipping on the outside (or inside) of a fixed circle of radius c. The equations are $x = (c \pm a)\cos\left(\frac{a}{c}u\right) \mp b\cos\left[\left(1 \pm \frac{a}{c}\right)u\right]$, $y = (c \pm a)\sin\left(\frac{a}{c}u\right) - b\sin\left[\left(1 \pm \frac{a}{c}\right)u\right]$, where u = the angle which the moving radius makes with the line of centers; take the upper sign for the epi- and the lower for the hypo-trochoid. The curve is called prolate or curtate according as $b < a$ or $b > a$. When $b = a$, the special case of the epi- or hypocycloid arises.

The Involute of a Circle is the curve traced by the end of a taut string which is unwound from the circumference of a fixed circle, of radius c. If QP



Involute of Circle.
FIG. 73.



Spiral of Archimedes.
FIG. 74.

is the free portion of the string at any instant (Fig. 73), QP will be tangent to the circle at Q, and the length of QP = length of arc QA; hence the construc-

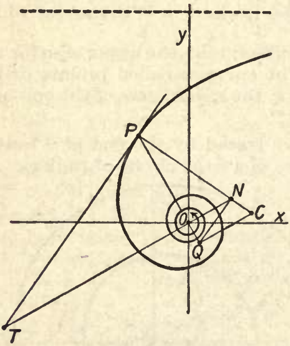
tion of the curve. The equations of the curve in parametric form (axes as in figure) are $x = c(\cos u + \text{rad } u \sin u)$, $y = c(\sin u - \text{rad } u \cos u)$, where rad u is the radian measure of the angle u which OQ makes with the x-axis. Length of arc $AP = \frac{1}{2}c(\text{rad } u)^2$; radius of curvature at P is QP.

The Spiral of Archimedes (Fig. 74) is traced by a point P which, starting from O, moves with uniform velocity along a ray OP, while the ray itself revolves with uniform angular velocity about O. Polar equation: $r = k \text{ rad } \theta$, or $r = a\,(\theta/360°)$. Here $a = 2\pi k$ = the distance, measured along a radius, from each coil to the next.

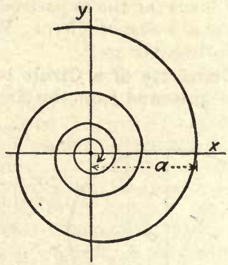
In order to construct the curve, draw radii $O1$, $O2$, $O3$, . . . making angles $\frac{1}{n}(360°)$, $\frac{2}{n}(360°)$, $\frac{3}{n}(360°)$, . . . with Ox, and along these radii lay off distances equal to $\frac{1}{n}a$, $\frac{2}{n}a$, $\frac{3}{n}a$, . . .; the points thus reached will lie on the spiral. The figure shows one-half of the curve, corresponding to positive values of θ.

Construction for tangent and normal: Let PT and PN be the tangent and normal at any point P, the line TON being perpendicular to OP. Then $OT = r^2/k$, and $ON = k$, where $k = a/(2\pi)$. Hence the construction.

The radius of curvature at P is $R = (k^2 + r^2)^{3/2}/(2k^2 + r^2)$. To construct the center of curvature, C, draw NQ perpendicular to PN and PQ perpendicular to OP; then OQ will meet PN in C. Length of arc $OP = \frac{1}{2}k\,[\text{rad } \theta \sqrt{1 + (\text{rad } \theta)^2} + \sinh^{-1}(\text{rad } \theta)]$. After many windings, arc $OP = \frac{1}{2}r^2/k$, approximately.



Hyperbolic Spiral.
FIG. 75.

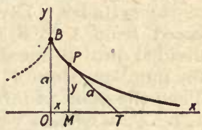


Logarithmic Spiral.
FIG. 76.

The Hyperbolic Spiral is the curve whose polar equation is $r = a/\text{rad } \theta$. To construct the curve, take a series of points along Ox (Fig. 75); through each of these points, with center at O, draw an arc extending into the upper half of the plane; and along each of these arcs lay off a length $= a$. The points thus reached will lie on the curve. A line parallel to the x-axis, at distance a, is an asymptote of the curve. The curve winds around and around the point O without ever reaching it (asymptotic point). The figure shows one-half of the curve, corresponding to positive values of θ. If PT and PN are the tangent and normal at any point P, the line TON being perpendicular to OP,

then $OT = a$, and $ON = r^2/a$. Hence a construction for the tangent and normal. Radius of curvature at P is $R = r/\sin^3 v$, where v = angle between OP and the tangent at P. Construction: At N draw a perpendicular to PN, meeting PO in Q; at Q draw a perpendicular to PQ, meeting PN in C; then C is the center of curvature for the point P.

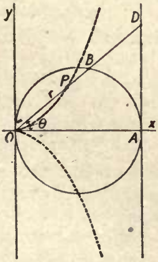
The Logarithmic Spiral (Fig. 76), is a curve which cuts the radii from O at a constant angle v, whose cotangent is m. Polar equation: $r = ae^{m \text{ rad } \theta}$. Here a is the value of r when $\theta = 0$. For large negative values of θ, the curve winds around O as an asymptotic point. If PT and PN are the tangent and normal at P, the line TON being perpendicular to OP (not shown in fig.), then $ON = rm$, and $PN = r\sqrt{1 + m^2} = r/\sin v$. Radius of curvature at P is PN. The evolute of the spiral is an equal spiral whose axis makes an angle $\frac{1}{2}\pi - (\log_e m)/m$ with the axis of the given spiral. Area swept out by the radius r from $r = 0$ (where $\theta = -\infty$) to $r = r$, is $A = r^2/(4m)$ = half the triangle OPT. Length of arc from O to $P = s = r/\cos v = PT$.



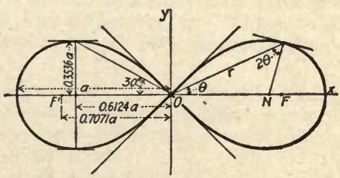
Tractrix.
FIG. 77.

The Tractrix, or Schiele's Anti-friction Curve (Fig. 77), is a curve such that the portion PT of the tangent between the point of contact and the x-axis is constant $= a$. Its equation is $x = \pm a \left[\cosh^{-1}\frac{a}{y} - \sqrt{1 - \left(\frac{y}{a}\right)^2}\right]$, or, in parametric form, $x = \pm a\,[t - \tanh t]$, $y = a/\cosh t$. (For tables of hyperbolic functions, see p. 60.) The x-axis is an asymptote of the curve. Length of arc $BP = a \log_e (a/y)$. The evolute (locus of centers of curvature) is the catenary whose lowest point is at B, and whose directrix is Ox.

The Cissoid (Fig. 78) is the locus of a point P such that OP, laid off on a variable ray from O, is equal to BD, the portion of the ray lying between a fixed circle through O and a fixed tangent at the point A opposite O. If a is the radius of the circle, the polar equation is $r = 2a \sin^2 \theta / \cos \theta$. Rectangular equation, $y^2(2a - x) = x^3$.



Cissoid.
FIG. 78.

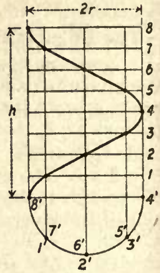


Lemniscate.
FIG. 79.

The Lemniscate (Fig. 79) is the locus of a point P the product of whose distances from two fixed points F, F' is constant, equal to $\frac{1}{2} a^2$. The distance $FF' = a\sqrt{2}$. Polar equation is $r = a\sqrt{\cos 2\theta}$. Angle between OP and the normal at P is 2θ. The two branches of the curve cross at right angles at O.

Maximum y occurs when $\theta = 30°$ and $r = a/\sqrt{2}$, and is equal to $\frac{1}{4} a\sqrt{2}$. Area of one loop $= a^2/2$.

The Helix (Fig. 80) is the curve of a screw thread on a cylinder of radius r. The curve crosses the elements of the cylinder at a constant angle, v. The pitch, h, is the distance between two coils of the helix, measured along an element of the cylinder; hence $h = 2\pi r \tan v$. Length of one coil $= \sqrt{(2\pi r)^2 + h^2} = 2\pi r/\cos v$. To construct the projection of a helix on a plane containing the axis of the cylinder, draw a rectangle, breadth $2r$ and height h, to represent the plane, with a semicircle below it, as in the figure, to represent the base of the cylinder. Divide h into equal parts (here 8), numbered from 1 to 8; think of the circumference as also divided into 8 equal parts, represented on the semicircle by numbers from $1'$ to $4'$ and back again from $4'$ to $8'$. Then the point of intersection of a horizontal line through 1, 2, . . . with a vertical line through $1'$, $2'$, . . . will be a point of the required projection. If the cylinder is rolled out on a plane, the development of the helix will be a straight line, with slope equal to $\tan v$.



Helix.
FIG. 80.

DIFFERENTIAL AND INTEGRAL CALCULUS

DERIVATIVES AND DIFFERENTIALS

Derivatives and Differentials. A **function** of a single variable x may be denoted by $f(x), F(x)$, etc. The value of the function when x has the value x_0 is then denoted by $f(x_0), F(x_0)$, etc. The **derivative** of a function $y = f(x)$ may be denoted by $f'(x)$, or by dy/dx. The value of the derivative at a given point $x = x_0$ is the **rate of change** of the function at that point; or, if the function is represented by a curve in the usual way (Fig. 1), the value of the derivative at any point shows the **slope of the curve** (that is, the slope of the tangent to the curve) at that point (positive if the tangent points upward, and negative if it points downward, moving to the right).

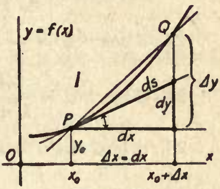


FIG. 1.

The **increment**, Δy (read: "delta y"), in y is the change produced in y by increasing x from x_0 to $x_0 + \Delta x$; that is, $\Delta y = f(x_0 + \Delta x) - f(x_0)$. The **differential**, dy, of y is the value which Δy would have if the curve coincided with its tangent. (The differential, dx, of x is the same as Δx when x is the independent variable.) Note that the derivative depends only on the value of x_0, while Δy and dy depend not only on x_0 but also on the value of Δx. The ratio $\Delta y/\Delta x$ represents the slope of the secant, and dy/dx the slope of the tangent (see Fig. 1). If Δx is made to approach zero, the secant approaches the tangent as a limiting position, so that the derivative $= f'(x) = \frac{dy}{dx} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}\right]$. Also, $dy = f'(x)\,dx$.

The symbol "lim" in connection with $\Delta x \doteq 0$ means "the limit, as Δx approaches 0, of ..." [A constant c is said to be the **limit** of a variable u if, whenever any quantity m has been assigned, there is a stage in the variation-process beyond which $|c - u|$ is always less than m; or, briefly, c is the limit of u if the difference between c and u can be made to become and remain as small as we please.]

To find the derivative of a given function at a given point: (1) If the function is given only by a curve, measure graphically the slope of the tangent at the point in question; (2) if the function is given by a mathematical expression, use the following rules for differentiation. These rules give, directly, the differential, dy, in terms of dx; to find the derivative, dy/dx, divide through by dx.

Rules for Differentiation. (Here $u, v, w, \ldots$ represent any functions of a variable x, or may themselves be independent variables. a is a constant which does not change in value in the same discussion; $e = 2.71828$.)

1. $d(a + u) = du$. 2. $d(au) = adu$.
3. $d(u + v + w + \ldots) = du + dv + dw + \ldots$
4. $d(uv) = udv + vdu$.
5. $d(uvw\ldots) = (uvw\ldots)\left(\frac{du}{u} + \frac{dv}{v} + \frac{dw}{w} + \ldots\right)$
6. $d\frac{u}{v} = \frac{vdu - udv}{v^2}$.
7. $d(u^m) = mu^{m-1}du$ when m is not $= -1$.

Thus, $d(u^2) = 2udu$; $d(u^3) = 3u^2du$; etc.

8. $d\sqrt{u} = \frac{du}{2\sqrt{u}}$.

9. $d\left(\frac{1}{u}\right) = -\frac{du}{u^2}$.

10. $d(e^u) = e^u du$.

11. $d(a^u) = (\log_e a)a^u du$.

12. $d \log_e u = \frac{du}{u}$.

13. $d \log_{10} u = (\log_{10} e)\frac{du}{u} = (0.4343\ldots)\frac{du}{u}$.

14. $d \sin u = \cos u\, du$.

15. $d \csc u = -\cot u \csc u\, du$.

16. $d \cos u = -\sin u\, du$.

17. $d \sec u = \tan u \sec u\, du$.

18. $d \tan u = \sec^2 u\, du$.

19. $d \cot u = -\csc^2 u\, du$.

20. $d \sin^{-1} u = \frac{du}{\sqrt{1-u^2}}$.

21. $d \csc^{-1} u = -\frac{du}{u\sqrt{u^2-1}}$.

22. $d \cos^{-1} u = -\frac{du}{\sqrt{1-u^2}}$.

23. $d \sec^{-1} u = \frac{du}{u\sqrt{u^2-1}}$.

24. $d \tan^{-1} u = \frac{du}{1+u^2}$.

25. $d \cot^{-1} u = -\frac{du}{1+u^2}$.

26. $d \log_e \sin u = \cot u\, du$.

27. $d \log_e \tan u = \frac{2du}{\sin 2u}$.

28. $d \log_e \cos u = -\tan u\, du$.

29. $d \log_e \cot u = -\frac{2du}{\sin 2u}$.

30. $d \sinh u = \cosh u\, du$.

31. $d\, \text{csch}\, u = -\text{csch}\, u \coth u\, du$.

32. $d \cosh u = \sinh u\, du$.

33. $d\, \text{sech}\, u = -\text{sech}\, u \tanh u\, du$.

34. $d \tanh u = \text{sech}^2 u\, du$.

35. $d \coth u = -\text{csch}^2 u\, du$.

36. $d \sinh^{-1} u = \frac{du}{\sqrt{u^2+1}}$.

37. $d\, \text{csch}^{-1} u = -\frac{du}{u\sqrt{u^2+1}}$.

38. $d \cosh^{-1} u = \frac{du}{\sqrt{u^2-1}}$.

39. $d\, \text{sech}^{-1} u = -\frac{du}{u\sqrt{1-u^2}}$.

40. $d \tanh^{-1} u = \frac{du}{1-u^2}$.

41. $d \coth^{-1} u = \frac{du}{1-u^2}$.

42. $d(u^v) = (u^{v-1})(u \log_e u\, dv + v\, du)$.

Derivatives of Higher Orders. The derivative of the derivative is called the second derivative; the derivative of this, the third derivative; and so on. Notation: if $y = f(x)$,

$$f'(x) = D_x y = \frac{dy}{dx}; \quad f''(x) = D_x^2 y = \frac{d^2y}{dx^2}; \quad f'''(x) = D_x^3 y = \frac{d^3y}{dx^3}; \quad \text{etc.}$$

NOTE. If the notation d^2y/dx^2 is used, this must not be treated as a fraction, like dy/dx but as an inseperable symbol, made up of a symbol of operation, d^2/dx^2, and an operand y

The geometric meaning of the second derivative is this: if the original function $y = f(x)$ is represented by a curve in the usual way, then at any point where $f''(x)$ is *positive*, the curve is *concave upward*, and at any point where $f''(x)$ is *negative*, the curve is *concave downward* (Fig. 2). When $f''(x) = 0$, the curve usually has a **point of inflection.**

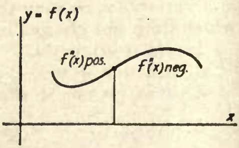


FIG. 2.

Differentials of Higher Orders. The differential of the differential is called the second differential; the differential of

this, the third differential; etc. These quantities are of little importance except in the case where dx = a constant. In this case

$$dy = f'(x)dx; \quad d^2y = f''(x)\cdot(dx)^2; \quad d^3y = f'''(x)\cdot(dx)^3; \ . \ . \ .$$

The first, second, third, etc., differentials are close approximations to the first, second, third, etc., differences (p. 115), and are therefore sometimes useful in constructing tables. Thus, denoting the first, second, third, etc., differences by D', D'', D''', etc., and, assuming always that dx = a constant,

$$D' = dy + \tfrac{1}{2} d^2y + \tfrac{1}{6} d^3y + \tfrac{1}{24} d^4y + \ . \ . \ . ; \quad d^3y = D''' - \tfrac{3}{2} D'''' + \ . \ . \ .$$
$$D'' = d^2y + d^3y + \tfrac{7}{12} d^4y + \ . \ . \ . ; \quad d^2y = D'' - D''' + \tfrac{11}{12} D'''' + \ . \ . \ .$$
$$D''' = d^3y + \tfrac{3}{2} d^4y + \ . \ . \ . ; \quad dy = D' - \tfrac{1}{2} D'' + \tfrac{1}{3} D''' - \tfrac{1}{4} D'''' + \ . \ . \ .$$

Functions of Two or More Variables may be denoted by $f(x, y, \ . \ . \ .)$, $F(x, y, \ . \ . \ .)$, etc. The derivative of such a function $u = f(x, y, \ . \ . \ .)$ formed on the assumption that x is the only variable (y, . . . being regarded for the moment as constants) is called the **partial derivative of u with respect to x**, and is denoted by $f_x(x,y)$, or D_xu, or $\dfrac{d_xu}{dx}$, or $\dfrac{\partial u}{\partial x}$. Similarly, the partial derivative of u with respect to y is $f_y(x,y)$, or D_yu, or $\dfrac{d_yu}{dy}$, or $\dfrac{\partial u}{\partial y}$.

NOTE. In the third notation, d_xu denotes the differential of u formed on the assumption that x is the only variable. If the fourth notation, $\partial u/\partial x$, is used, this must not be treated as a fraction like du/dx; the $\partial/\partial x$ is a symbol of operation, operating on u, and the "∂x" must not be separated.

Partial derivatives of the second order are denoted by f_{xx}, f_{xy}, f_{yy}, or by D_x^2u, $D_x(D_yu)$, D_y^2u, or by $\dfrac{\partial^2u}{\partial x^2}$, $\dfrac{\partial^2u}{\partial x\partial y}$, $\dfrac{\partial^2u}{\partial y^2}$, the last symbols being "inseparable." Similarly for higher derivatives. Note that $f_{xy} = f_{yx}$.

If increments Δx, Δy, (or dx, dy) are assigned to the independent variables x, y, the increment, Δu, produced in $u = f(x,y)$ is

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x,y);$$

while the **differential**, du, that is, the value which Δu would have if the partial derivatives of u with respect to x and y were constant, is given by

$$du = (f_x)\cdot dx + (f_y)\cdot dy.$$

Here the coefficients of dx and dy are the values of the partial derivatives of u at the point in question.

If x and y are functions of a third variable t, then the equation

$$\frac{du}{dt} = (f_x)\,\frac{dx}{dt} + (f_y)\,\frac{dy}{dt}$$

expresses the rate of change of u with respect to t, in terms of the separate rates of change of x and y with respect to t.

For the graphical representation of $u = f(x,y)$, see p. 178.

Implicit Functions. If $f(x,y) = 0$, either of the variables x and y is said to be an implicit function of the other. To find dy/dx, either (1) solve for y in terms of x, and then find dy/dx directly; or (2) differentiate the equation through as it stands, remembering that both x and y are variables, and then divide by dx; or (3) use the formula $dy/dx = -(f_x/f_y)$, where f_x and f_y are the partial derivatives of $f(x,y)$ at the point in question.

MAXIMA AND MINIMA

A Function of One Variable, as $y = f(x)$, is said to have a **maximum** at a point $x = x_0$, if at that point the slope of the curve is zero and the concavity

downward (see Fig. 3); a sufficient condition for a maximum is $f'(x_0) = 0$ and $f''(x_0)$ negative. Similarly, $f(x)$ has a **minimum** if the slope is zero and the concavity upward; a sufficient condition for a minimum is $f'(x_0) = 0$ and $f''(x_0)$ positive. If $f'(x_0) = 0$ and $f''(x_0) = 0$ and $f'''(x_0) \neq 0$, the point x_0 will be a **point of inflection.** If $f'(x_0) = 0$ and $f''(x_0) = 0$ and $f'''(x_0) = 0$, the point x_0 will be a maximum if $f''''(x_0) < 0$, and a minimum if $f''''(x_0) > 0$. It is usually sufficient, however, in any practical case, to find the values of x which make $f'(x) = 0$, and then decide, from a general knowledge of the curve, which of these values (if any) give maxima or minima, without investigating the higher derivatives.

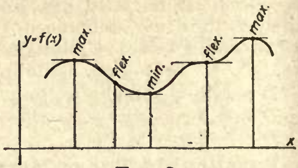


FIG. 3.

A Function of Two Variables, as $u = f(x,y)$, will have a **maximum** at a point (x_0,y_0) if at that point $f_x = 0$, $f_y = 0$, and $f_{xx} < 0$, $f_{yy} < 0$; and a **minimum** if at that point $f_x = 0$, $f_y = 0$, and $f_{xx} > 0$, $f_{yy} > 0$; provided, in each case, $(f_{xx})(f_{yy}) - (f_{xy})^2$ is positive. If $f_x = 0$ and $f_y = 0$, and f_{xx} and f_{yy} have opposite signs, the point (x_0,y_0) will be a "saddle point" of the surface representing the function (p. 178).

EXPANSION IN SERIES

The range of values of x for which each of the series is convergent is stated at the right of the series.

Arithmetical and Geometrical Series, and the **Binomial Theorem.** See p. 114.

Exponential and Logarithmic Series.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots; \qquad -\infty < x < +\infty.$$

$$a^x = e^{mx} = 1 + \frac{m}{1!}x + \frac{m^2}{2!}x^2 + \frac{m^3}{3!}x^3 + \cdots; \quad a > 0, \quad -\infty < x < +\infty,$$

where $m = \log_e a = (2.3026)(\log_{10} a)$.

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots; \qquad -1 < x < +1.$$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots; \qquad -1 < x < +1.$$

$$\log_e\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right); \qquad -1 < x < +1.$$

$$\log_e\left(\frac{x+1}{x-1}\right) = 2\left(\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \frac{1}{7x^7} + \cdots\right); \qquad x < -1 \text{ or } +1 < x.$$

$$\log_e x = 2\left[\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right]; \qquad 0 < x < \infty.$$

$$\log_e(a+x) = \log_e a + 2\left[\frac{x}{2a+x} + \frac{1}{3}\left(\frac{x}{2a+x}\right)^3 + \frac{1}{5}\left(\frac{x}{2a+x}\right)^5 + \cdots\right]; \qquad \begin{cases} 0 < a < +\infty \\ -a < x < +\infty \end{cases}$$

Series for the Trigonometric Functions. In the following formulæ, *all angles must be expressed in radians.* If D = the number of degrees in the angle, and x = its radian measure, then $x = 0.017453\ D$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots; \qquad -\infty < x < +\infty.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots; \qquad -\infty < x < +\infty.$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \ldots; \qquad -\pi/2 < x < +\pi/2.$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \ldots; \qquad -\pi < x < +\pi.$$

$$\sin^{-1} y = y + \frac{y^3}{6} + \frac{3y^5}{40} + \frac{5y^7}{112} + \ldots; \qquad -1 \leqq y \leqq +1.$$

$$\tan^{-1} y = y - \frac{y^3}{3} + \frac{y^5}{5} - \frac{y^7}{7} + \ldots; \qquad -1 \leqq y \leqq +1.$$

$\cos^{-1} y = \frac{1}{2}\pi - \sin^{-1} y$; $\quad \cot^{-1} y = \frac{1}{2}\pi - \tan^{-1} y$.

Series for the Hyperbolic Functions (x a pure number).

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots; \qquad -\infty < x < \infty.$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots; \qquad -\infty < x < \infty.$$

$$\sinh^{-1} y = y - \frac{y^3}{6} + \frac{3y^5}{40} - \frac{5y^7}{112} + \ldots; \qquad -1 < y < +1.$$

$$\tanh^{-1} y = y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \ldots; \qquad -1 < y < +1.$$

General Formulæ of Maclaurin and Taylor. If $f(x)$ and all its derivatives are continuous in the neighborhood of the point $x = 0$ (or $x = a$), then, for any value of x in this neighborhood, the function $f(x)$ may be expressed as a power series arranged according to ascending powers of x (or of $x - a$), as follows:

$$(1)\quad f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + (P_n)x^n. \quad \text{(Maclaurin.)}$$

$$(2)\quad f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + (Q_n)(x-a)^n. \quad \text{(Taylor.)}$$

Here $(P_n)x^n$, or $(Q_n)(x - a)^n$, is called the **remainder term**; the values of the coefficients P_n and Q_n may be expressed as follows:

$P_n = \{f^{(n)}(sx)\}/n! = \{(1 - t)^{n-1}\ f^{(n)}(tx)\}/(n - 1)!$

$Q_n = \{f^{(n)}[a + s(x - a)]\}/n! = \{(1 - t)^{n-1} f^{(n)}[a + t(x - a)]\}/(n - 1)!$

where s and t are certain unknown numbers between 0 and 1; the s-form is due to Lagrange, the t-form to Cauchy.

The error due to neglecting the remainder term is less than $(\bar{P}_n)x^n$, or

$(\bar{Q}_n)(x - a)^n$, where $\bar{P}_n$, or $\bar{Q}_n$, is the largest value taken on by P_n, or Q_n, when s or t ranges from 0 to 1. If this error, which depends on both n and x, approaches 0 as n increases (for any given value of x), then the general-expression-with-remainder becomes (for that value of x) a convergent infinite series.

The sum of the first few terms of Maclaurin's series gives a good approximation to $f(x)$ for values of x near $x = 0$; Taylor's series gives a similar approximation for values near $x = a$.

Fourier's Series. Let $f(x)$ be a function which is finite in the interval from $x = -c$ to $x = +c$ and has only a finite number of discontinuities in that interval (see note below), and only a finite number of maxima and minima. Then, for any value of x between $-c$ and c,

$$f(x) = \tfrac{1}{2} a_0 + a_1 \cos \frac{\pi x}{c} + a_2 \cos \frac{2\pi x}{c} + a_3 \cos \frac{3\pi x}{c} + \ldots$$
$$+ b_1 \sin \frac{\pi x}{c} + b_2 \sin \frac{2\pi x}{c} + b_3 \sin \frac{3\pi x}{c} + \ldots$$

where the constant coefficients are determined as follows:

$$a_n = \frac{1}{c}\int_{-c}^{c} f(t) \cos \frac{n\pi t}{c}\, dt, \quad b_n = \frac{1}{c}\int_{-c}^{c} f(t) \sin \frac{n\pi t}{c}\, dt.$$

In case the curve $y = f(x)$ is symmetrical with respect to the origin, the a's are all zero, and the series is a sine series. In case the curve is symmetrical with respect to the y-axis, the b's are all zero, and a cosine series results. (In this case, the series will be valid not only for values of x between $-c$ and c, but also for $x = -c$ and $x = c$.) A Fourier's series can be integrated term by term; but the result of differentiating term by term will in general not be a convergent series.

NOTE. If $x = x_0$ is a point of discontinuity, $f(x_0)$ is to be defined as $\tfrac{1}{2}[f_1(x_0) + f_2(x_0)]$, where $f_1(x_0)$ is the limit of $f(x)$ when x approaches x_0 from below, and $f_2(x_0)$ is the limit of $f(x)$ when x approaches x_0 from above.

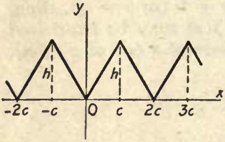


FIG. 4.

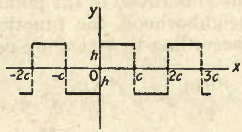


FIG. 5.

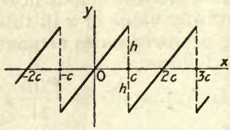


FIG. 6.

Examples of Fourier's Series.

1. If $y = f(x)$ is the curve in Fig. 4,

$$y = \frac{h}{2} - \frac{4h}{\pi^2}\left(\cos\frac{\pi x}{c} + \frac{1}{9}\cos\frac{3\pi x}{c} + \frac{1}{25}\cos\frac{5\pi x}{c} + \ldots\right)$$

2. If $y = f(x)$ is the curve in Fig. 5,

$$y = \frac{4h}{\pi}\left(\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right)$$

3. If $y = f(x)$ is the curve in Fig. 6,

$$y = \frac{2h}{\pi}\left(\sin\frac{\pi x}{c} - \frac{1}{2}\sin\frac{2\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} - \ldots\right)$$

INDETERMINATE FORMS

In the following paragraphs, $f(x)$, $g(x)$ denote functions which approach 0; $F(x)$, $G(x)$ functions which increase indefinitely; and $U(x)$ a function which approaches 1; when x approaches a definite quantity a. The problem in each case is to find the limit approached by certain combinations of these functions when x approaches a. The symbol $\doteq$ is to be read "approaches."

CASE 1. "$\frac{0}{0}$." To find the limit of $f(x)/g(x)$ when $f(x) \doteq 0$ and $g(x) \doteq 0$, use the theorem that $\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$, where $f'(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$. This second limit may be easier to find than the first. If $f'(x) \doteq 0$ and $g'(x) \doteq 0$, apply the same theorem a second time: $\lim \frac{f'(x)}{g'(x)} = \lim \frac{f''(x)}{g''(x)}$; and so on.

CASE 2. "$\frac{\infty}{\infty}$." If $F(x) \doteq \infty$ and $G(x) \doteq \infty$, then $\lim \frac{F(x)}{G(x)} = \lim \frac{F'(x)}{G'(x)}$, precisely as in Case 1.

CASE 3. "$0 \cdot \infty$." To find the limit of $f(x) \cdot F(x)$ when $f(x) \doteq 0$ and $F(x) \doteq \infty$, write $\lim [f(x) \cdot F(x)] = \lim \frac{f(x)}{1/F(x)}$, or $= \lim \frac{F(x)}{1/f(x)}$; then proceed as in Case 1 or Case 2.

CASE 4. "0^0." If $f(x) \doteq 0$ and $g(x) \doteq 0$, find $\lim [f(x)]^{g(x)}$ as follows: let $y = [f(x)]^{g(x)}$, and take the logarithm of both sides thus:

$$\log_e y = g(x) \log_e f(x);$$

next, find $\lim [g(x) \log_e f(x)]$, $= m$, by Case 3; then $\lim y = e^m$.

CASE 5. "1^∞." If $U(x) \doteq 1$ and $F(x) \doteq \infty$, find $\lim [U(x)]^{F(x)}$ as follows: let $y = [U(x)]^{F(x)}$, and take the logarithm of both sides, as in Case 4.

CASE 6. "∞^0." If $F(x) \doteq \infty$ and $f(x) \doteq 0$, find $\lim [F(x)]^{f(x)}$ as follows: let $y = [F(x)]^{f(x)}$, and take the logarithm of both sides, as in Case 4.

CASE 7. "$\infty - \infty$." If $F(x) \doteq \infty$ and $G(x) \doteq \infty$, write $\lim [F(x) - G(x)]$ $= \lim \frac{\frac{1}{G(x)} - \frac{1}{F(x)}}{\frac{1}{F(x) \cdot G(x)}}$; then proceed as in Case 1. Sometimes it is shorter to expand the functions in series. It should be carefully noticed that expressions like $0/0$, ∞/∞, etc., do not represent mathematical quantities.

CURVATURE

The **radius of curvature** R of a plane curve at any point P (Fig. 7) is the distance, measured along the normal, on the concave side of the curve, to the **center of curvature**, C, this point being the limiting position of the point of intersection of the normals at P and a neighboring point Q, as Q is made to approach P along the curve. If the equation of the curve is $y = f(x)$,

$$R = \frac{ds}{du} = \frac{[1 + (y')^2]^{3/2}}{y''}$$

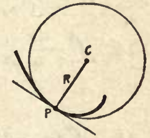


FIG. 7.

where $ds = \sqrt{dx^2 + dy^2}$ = the differential of arc, $u = \tan^{-1}[f'(x)]$ = the angle which the tangent at P makes with the x-axis, and $y' = f'(x)$ and $y'' = f''(x)$ are the first and second derivatives of $f(x)$ at the point P. Note that $dx = ds \cos u$ and $dy = ds \sin u$. The **curvature**, K, at the point P, is $K = 1/R = du/ds$; that is, the curvature is the rate at which the angle u is changing with respect to the length of arc s. If the slope of the curve is small, $K \approx f''(x)$.

If the equation of the curve in polar co-ordinates is $r = f(\theta)$, where r = radius vector and θ = polar angle, then

$$R = \frac{[r^2 + (r')^2]^{3/2}}{r^2 - rr'' + 2(r')^2},$$

where $r' = f'(\theta)$ and $r'' = f''(\theta)$.

The **evolute** of a curve is the locus of its centers of curvature. If one curve is the evolute of another, the second is called the **involute** of the first.

INDEFINITE INTEGRALS

An integral of $f(x)dx$ is any function whose differential is $f(x)dx$, and is denoted by $\int f(x)dx$. All the integrals of $f(x)dx$ are included in the expression $\int f(x)dx + C$, where $\int f(x)dx$ is any particular integral, and C is an arbitrary constant. The process of finding (when possible) an integral of a given function consists in recognizing by inspection a function which, when differentiated, will produce the given function; or in transforming the given function into a form in which such recognition is easy. The most common integrable forms are collected in the following brief table; for a more extended list, see B. O. Peirce's "Table of Integrals" (Ginn & Co.).

General Formulæ

1. $\int a\,du = a\int du = au + C$ 2. $\int (u + v)dx = \int u\,dx + \int v\,dx$

3. $\int u\,dv = uv - \int v\,du$ 4. $\int f(x)dx = \int f[F(y)]F'(y)dy,\ x = F(y)$

5. $\int dy \int f(x,y)dx = \int dx \int f(x,y)dy.$

Fundamental Integrals

6. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C$; when $n \neq -1$

7. $\int \dfrac{dx}{x} = \log_e x + C = \log_e cx$ 8. $\int e^x dx = e^x + C$

9. $\int \sin x\,dx = -\cos x + C$ 10. $\int \cos x\,dx = \sin x + C$

11. $\int \dfrac{dx}{\sin^2 x} = -\cot x + C$ 12. $\int \dfrac{dx}{\cos^2 x} = \tan x + C$

13. $\int \dfrac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C = -\cos^{-1} x + c$

14. $\int \dfrac{dx}{1 + x^2} = \tan^{-1} x + C = -\cot^{-1} x + c$

Rational Functions

15. $\int (a + bx)^n dx = \dfrac{(a + bx)^{n+1}}{(n + 1)b} + C$

16. $\int \frac{dx}{a + bx} = \frac{1}{b} \log_e (a + bx) + C = \frac{1}{b} \log_e c(a + bx)$

17. $\int \frac{1}{x^2} dx = -\frac{1}{x} + C$ 18. $\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)} + C$

19. $\int \frac{dx}{1 - x^2} = \frac{1}{2} \log_e \frac{1 + x}{1 - x} + C = \tanh^{-1} x + C,$ when $x < 1$

20. $\int \frac{dx}{x^2 - 1} = \frac{1}{2} \log_e \frac{x - 1}{x + 1} + C = -\coth^{-1} x + C,$ when $x > 1$

$$\left.\begin{aligned} &21.\ \int \frac{dx}{a + bx^2} = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\sqrt{\frac{b}{a}}\, x\right) + C \\ &22.\ \int \frac{dx}{a - bx^2} = \frac{1}{2\sqrt{ab}} \log_e \frac{\sqrt{ab} + bx}{\sqrt{ab} - bx} + C \\ &\qquad = \frac{1}{\sqrt{ab}} \tanh^{-1}\left(\sqrt{\frac{b}{a}}\, x\right) + C \end{aligned}\right\} \text{when } a > 0,\ b > 0$$

23. $$\int \frac{dx}{a + 2bx + cx^2} = \frac{1}{\sqrt{ac - b^2}} \tan^{-1} \frac{b + cx}{\sqrt{ac - b^2}} + C \quad \text{when } ac - b^2 > 0;$$

$$\left.\begin{aligned} &= \frac{1}{2\sqrt{b^2 - ac}} \log_e \frac{\sqrt{b^2 - ac} - b - cx}{\sqrt{b^2 - ac} + b + cx} + C \\ &= -\frac{1}{\sqrt{b^2 - ac}} \tanh^{-1} \frac{b + cx}{\sqrt{b^2 - ac}} + C, \end{aligned}\right\} \text{when } b^2 - ac > 0;$$

24. $\int \frac{dx}{a + 2bx + cx^2} = -\frac{1}{b + cx} + C,$ when $b^2 = ac$

25. $\int \frac{(m + nx)dx}{a + 2bx + cx^2} = \frac{n}{2c} \log_e (a + 2bx + cx^2) + \frac{mc - nb}{c} \int \frac{dx}{a + 2bx + cx^2}$

26. In $\int \frac{f(x)dx}{a + 2bx + cx^2}$, if $f(x)$ is a polynominal of higher than the first degree, divide by the denominator before integrating.

27. $$\int \frac{dx}{(a + 2bx + cx^2)^p} = \frac{1}{2(ac - b^2)(p - 1)} \times \frac{b + cx}{(a + 2bx + cx^2)^{p-1}} + \frac{(2p - 3)c}{2(ac - b^2)(p - 1)} \int \frac{dx}{(a + 2bx + cx^2)^{p-1}}$$

28. $$\int \frac{(m + nx)dx}{(a + 2bx + cx^2)^p} = -\frac{n}{2c(p - 1)} \times \frac{1}{(a + 2bx + cx^2)^{p-1}} + \frac{mc - nb}{c} \int \frac{dx}{(a + 2bx + cx^2)^p}$$

29. $$\int x^{m-1}(a + bx)^n dx = \frac{x^{m-1}(a + bx)^{n+1}}{(m + n)b} - \frac{(m - 1)a}{(m + n)b} \int x^{m-2}(a + bx)^n dx$$

$$= \frac{x^m(a + bx)^n}{m + n} + \frac{na}{m + n} \int x^{m-1}(a + bx)^{n-1} dx$$

Irrational Functions

30. $\int \sqrt{a + bx}\, dx = \frac{2}{3b}\left(\sqrt{a + bx}\right)^3 + C$

31. $\int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b}\sqrt{a + bx} + C$

32. $\int \frac{(m + nx)dx}{\sqrt{a + bx}} = \frac{2}{3b^2}(3mb - 2an + nbx)\sqrt{a + bx} + C$

33. $\int \frac{dx}{(m + nx)\sqrt{a + bx}}$; substitute $y = \sqrt{a + bx}$, and use 21 and 22

34. $\int \frac{f(x, \sqrt[n]{a + bx})}{F(x, \sqrt[n]{a + bx})}dx$; substitute $\sqrt[n]{a + bx} = y$

35. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C = -\cos^{-1}\frac{x}{a} + c$

36. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log_e\left[x + \sqrt{a^2 + x^2}\right] + C = \sinh^{-1}\frac{x}{a} + c$

37. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e\left[x + \sqrt{x^2 - a^2}\right] + C = \cosh^{-1}\frac{x}{a} + c$

38. $\int \frac{dx}{\sqrt{a + 2bx + cx^2}} = \frac{1}{\sqrt{c}}\log_e\left[b + cx + \sqrt{c}\sqrt{a + 2bx + cx^2}\right] + C,$ when $c > 0$;

$$= \frac{1}{\sqrt{c}}\sinh^{-1}\frac{b + cx}{\sqrt{ac - b^2}} + C, \quad \text{when } ac - b^2 > 0;$$

$$= \frac{1}{\sqrt{c}}\cosh^{-1}\frac{b + cx}{\sqrt{b^2 - ac}} + C, \quad \text{when } b^2 - ac > 0;$$

$$= \frac{-1}{\sqrt{-c}}\sin^{-1}\frac{b + cx}{\sqrt{b^2 - ac}} + C, \quad \text{when } c < 0$$

39. $\int \frac{(m + nx)dx}{\sqrt{a + 2bx + cx^2}} = \frac{n}{c}\sqrt{a + 2bx + cx^2}$

$$+ \frac{mc - nb}{c}\int \frac{dx}{\sqrt{a + 2bx + cx^2}}$$

40. $\int \frac{x^m dx}{\sqrt{a + 2bx + cx^2}} = \frac{x^{m-1}X}{mc} - \frac{(m - 1)a}{mc}\int \frac{x^{m-2}\,dx}{X}$

$$- \frac{(2m - 1)b}{mc}\int \frac{x^{m-1}\,dx}{X}, \quad \text{where } X = \sqrt{a + 2bx + cx^2}$$

41. $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log_e\left(x + \sqrt{a^2 + x^2}\right) + C$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c$$

42. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

43. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e \left(x + \sqrt{x^2 - a^2}\right) + C$

$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\frac{x}{a} + c$

44. $\int \sqrt{a + 2bx + cx^2}\, dx = \frac{b + cx}{2c}\sqrt{a + 2bx + cx^2}$

$+ \frac{ac - b^2}{2c}\int \frac{dx}{\sqrt{a + 2bx + cx^2}} + C$

Transcendental Functions

45. $\int a^x dx = \frac{a^x}{\log_e a} + C$

46. $\int x^n e^{ax}\, dx = \frac{x^n e^{ax}}{a}\left[1 - \frac{n}{ax} + \frac{n(n-1)}{a^2x^2} - \ldots \pm \frac{n!}{a^n x^n}\right] + C$

47. $\int \log_e x\, dx = x \log_e x - x + C$

48. $\int \frac{\log_e x}{x^2}\, dx = -\frac{\log_e x}{x} - \frac{1}{x} + C$

49. $\int \frac{(\log_e x)^n}{x}\, dx = \frac{1}{n+1}(\log_e x)^{n+1} + C$

50. $\int \sin^2 x\, dx = -\tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + C = -\tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + C$

51. $\int \cos^2 x\, dx = \tfrac{1}{4}\sin 2x + \tfrac{1}{2}x + C = \tfrac{1}{2}\sin x \cos x + \tfrac{1}{2}x + C$

52. $\int \sin mx\, dx = -\frac{\cos mx}{m} + C$ 53. $\int \cos mx\, dx = \frac{\sin mx}{m} + C$

54. $\int \sin mx \cos nx\, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C$

55. $\int \sin mx \sin nx\, dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)} + C$

56. $\int \cos mx \cos nx\, dx = \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)} + C$

57. $\int \tan x\, dx = -\log_e \cos x + C$ 58. $\int \cot x\, dx = \log_e \sin x + C$

59. $\int \frac{dx}{\sin x} = \log_e \tan \frac{x}{2} + C$ 60. $\int \frac{dx}{\cos x} = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$

61. $\int \frac{dx}{1 + \cos x} = \tan\frac{x}{2} + C$ 62. $\int \frac{dx}{1 - \cos x} = -\cot\frac{x}{2} + C$

63. $\int \sin x \cos x\, dx = \tfrac{1}{2}\sin^2 x + C$ 64. $\int \frac{dx}{\sin x \cos x} = \log_e \tan x + C$

65.* $\int \sin^n x\, dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

66.* $\int \cos^n x\, dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

* If n is an odd number, substitute $\cos x = z$ or $\sin x = z$.

67. $\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

68. $\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx$

69. $\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}$

70. $\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}$

71.* $\int \sin^p x \cos^q x\,dx = \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sin^p x \cos^{q-2} x\,dx$

$= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\,dx$

72.* $\int \sin^{-p} x \cos^q x\,dx = -\frac{\sin^{-p+1} x \cos^{q+1} x}{p-1} + \frac{p-q-2}{p-1}\int \sin^{-p+2} x \cos^q x\,dx$

73.* $\int \sin^p x \cos^{-q} x\,dx = \frac{\sin^{p+1} x \cos^{-q+1} x}{q-1} + \frac{q-p-2}{q-1}\int \sin^p x \cos^{-q+2} x\,dx$

74. $\int \frac{dx}{a + b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1}\left(\sqrt{\frac{a-b}{a+b}} \tan \tfrac{1}{2}x\right) + C$, when $a^2 > b^2$

$$\left.\begin{aligned} &= \frac{1}{\sqrt{b^2-a^2}} \log_e \frac{b + a\cos x + \sin x\sqrt{b^2-a^2}}{a + b\cos x} + C, \\ &= \frac{2}{\sqrt{b^2-a^2}} \tanh^{-1}\left(\sqrt{\frac{b-a}{b+a}} \tan \tfrac{1}{2}x\right) + C, \end{aligned}\right\} \text{when } a^2 < b^2$$

75. $\int \frac{\cos x\,dx}{a + b\cos x} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a + b\cos x} + C$

76. $\int \frac{\sin x\,dx}{a + b\cos x} = -\frac{1}{b}\log_e (a + b\cos x) + C$

77. $\int \frac{A + B\cos x + C\sin x}{a + b\cos x + c\sin x}\,dx = A\int \frac{dy}{a + p\cos y}$

$+ (B\cos u + C\sin u)\int \frac{\cos y\,dy}{a + p\cos y} - (B\sin u - C\cos u)\int \frac{\sin y\,dy}{a + p\cos y},$

where $b = p\cos u$, $c = p\sin u$ and $x - u = y$.

78. $\int e^{ax} \sin bx\,dx = \frac{a\sin bx - b\cos bx}{a^2+b^2} e^{ax} + C$

79. $\int e^{ax} \cos bx\,dx = \frac{a\cos bx + b\sin bx}{a^2+b^2} e^{ax} + C$

80. $\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$

81. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1-x^2} + C$

82. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \tfrac{1}{2}\log_e (1+x^2) + C$

83. $\int \cot^{-1} x\,dx = x\cot^{-1} x + \tfrac{1}{2}\log_e (1+x^2) + C$

* If p or q is an odd number, substitute $\cos x = z$ or $\sin x = z$.

84. $\int \sinh x\,dx = \cosh x + C$ 85. $\int \tanh x\,dx = \log_e \cosh x + C$

86. $\int \cosh x\,dx = \sinh x + C$ 87. $\int \coth x\,dx = \log_e \sinh x + C$

88. $\int \operatorname{sech} x\,dx = 2 \tan^{-1}(e^x) + C$ 89. $\int \operatorname{csch} x\,dx = \log_e \tanh(x/2) + C$

90. $\int \sinh^2 x\,dx = \tfrac{1}{2} \sinh x \cosh x - \tfrac{1}{2}x + C$

91. $\int \cosh^2 x\,dx = \tfrac{1}{2} \sinh x \cosh x + \tfrac{1}{2}x + C$

92. $\int \operatorname{sech}^2 x\,dx = \tanh x + C$ 93. $\int \operatorname{csch}^2 x\,dx = -\coth x + C$

DEFINITE INTEGRALS

The definite integral of $f(x)dx$ from $x = a$ to $x = b$, denoted by $\int_a^b f(x)dx$, is the limit (as n increases indefinitely) of a sum of n terms:

$$\int_a^b f(x)dx = \lim_{n=\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x],$$

built up as follows: Divide the interval from a to b into n equal parts, and call each part Δx, $= (b - a)/n$; in each of these intervals take a value of x (say $x_1, x_2, \ldots x_n$), find the value of the function $f(x)$ at each of these points, and multiply it by Δx, the width of the interval; then take the limit of the sum of the terms thus formed, when the number of terms increases indefinitely, while each individual term approaches zero.

Geometrically, $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ (Fig. 8); that is, briefly, the "area under the curve, from a to b." The **fundamental theorem** for the evaluation of a definite integral is the following:

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_{x=b} - \left[\int f(x)dx\right]_{x=a};$$

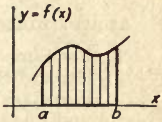


FIG. 8.

that is, the definite integral is equal to the difference between two values of any one of the indefinite integrals of the function in question. In other words, the limit of a sum can be found whenever the function can be integrated.

Properties of Definite Integrals.

$$\int_a^b = -\int_b^a; \quad \int_a^c + \int_c^b = \int_a^b.$$

THE MEAN-VALUE THEOREM FOR INTEGRALS.

$$\int_a^b F(x)\, f(x)dx = F(X)\int_a^b f(x)dx,$$

provided $f(x)$ does not change sign from $x = a$ to $x = b$; here X is some (unknown) value of x intermediate between a and b.

THEOREM ON CHANGE OF VARIABLE. In evaluating $\int_{x=a}^{x=b} f(x)dx$, $f(x)dx$ may be replaced by its value in terms of a new variable t and dt, and $x = a$ and $x = b$ by the corresponding values of t, provided that throughout the interval the relation between x and t is a one-to-one correspondence (that is, to each value of x there corresponds one and only one value of t, and to each value of t there corresponds one and only one value of x).

DIFFERENTIATION WITH RESPECT TO THE UPPER LIMIT. If b is variable, then $\int_a^b f(x)dx$ is a function of b, whose derivative is

$$\frac{d}{db}\int_a^b f(x)dx = f(b).$$

DIFFERENTIATION WITH RESPECT TO A PARAMETER.

$$\frac{\partial}{\partial c}\int_a^b f(x,c)dx = \int_a^b \frac{\partial f(x,c)}{\partial c}dx.$$

Functions Defined by Definite Integrals. The following definite integrals have received special names, and their values have been tabulated; see, for example, B. O. Peirce's "Table of Integrals."

1. Elliptic integral of the first kind $= F(u, k) = \int_0^u \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} (k^2 < 1)$

2. Elliptic integral of the second kind $= E(u, k) = \int_0^u \sqrt{1 - k^2 \sin^2 x}\, dx$ $(k^2 < 1)$

3, 4. Complete elliptic integrals of the first and second kinds; put $u = \pi/2$ in (1) and (2).

5. The Probability integral $= \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$

6. The Gamma function $= \Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

Approximate Methods of Integration. Mechanical Quadrature.

(1) Use Simpson's rule. See p. 106.

(2) Expand the function in a power series, and integrate term by term.

(3) Plot the area under the curve $y = f(x)$ from $x = a$ to $x = b$ on squared paper and measure this area roughly by "counting squares," or more accurately, by the use of a planimeter ($14 to $35; instruction for use with each instrument).

(4) Coradi's Mechanical Integraph ($240) provides a means of drawing on paper the curve $y = \int f(x)dx$, when the curve $y = f(x)$ is given, and can be used to facilitate the solution of certain differential equations. Full instructions for use with each instrument.

Double Integrals. The notation $\int\int f(x, y)dy\, dx$ means $\int\{\int f(x, y)dy\}dx$, the limits of integration in the inner, or first, integral being functions of x (or constants).

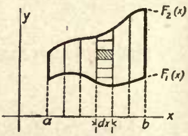


FIG. 9.

EXAMPLE. To find the weight of a plane area whose density, w, is variable, say $w = f(x, y)$. The weight of a typical element, $dx\,dy$, is $f(x, y)dx\,dy$. Keeping x and dx constant, and summing these elements from, say, $y = F_1(x)$ to $y = F_2(x)$, as determined by the shape of the boundary, the weight of a typical strip perpendicular to the x-axis is $dx\int_{y=F_1(x)}^{y=F_2(x)} f(x,y)dy$. Finally, summing these strips from, say, $x = a$ to $x = b$, the weight of the whole area is $\int_{x=a}^{x=b}\{dx\int_{y=F_1(x)}^{y=F_2(x)} f(x, y)dy\}$, or, briefly, $\int\int f(x, y)dy\,dx$.

DIFFERENTIAL EQUATIONS

An **ordinary differential equation** is one which contains a single independent variable, or argument, and a single dependent variable, or function, with its derivatives of various orders. A **partial differential equation** is one which contains a function of several independent variables, and its partial derivatives of various orders. The order of a differential equation is the order of the highest derivative which occurs in it. A solution of a differential equation is any relation between the variables, which, when substituted in the given equation, will satisfy it. The general solution of an ordinary differential equation of the nth order will contain n arbitrary constants. A differential equation is usually said to be solved when the problem is reduced to a simple quadrature, that is, an integration of the form $y = \int f(x)dx$.

Methods of Solving Ordinary Differential Equations

Differential Equations of the First Order

(1) If possible, separate the variables; that is, collect all the x's and dx on one side, and all the y's and dy on the other side; then integrate both sides, and add the constant of integration.

(2) If the equation is homogeneous in x and y, the value of dy/dx in terms of x and y will be of the form $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$. Substituting $y = xt$ will enable the variables to be separated. Solution: $\log_e x = \displaystyle\int \frac{dt}{f(t) - t} + C$.

(3) The expression $f(x,y)dx + F(x,y)dy$ is an *exact differential* if $\dfrac{\partial f(x,y)}{\partial y} = \dfrac{\partial F(x,y)}{\partial x}(= P$, say). In this case the solution of $f(x,y)dx + F(x,y)dy = 0$ is

$$\int f(x,y)dx + \int [F(x,y) - \int P dx]dy = C$$

or

$$\int F(x,y)dy + \int [f(x,y) - \int P dy]dx = C$$

(4) Linear differential equation of the first order: $\dfrac{dy}{dx} + f(x)\cdot y = F(x)$.

Solution: $y = e^{-P}\left\{\displaystyle\int e^P F(x)dx + C\right\}$, where $P = \int f(x)dx$.

(5) Bernoulli's equation: $\dfrac{dy}{dx} + f(x)\cdot y = F(x)\cdot y^n$. Substituting $y^{1-n} = v$ gives $\dfrac{dv}{dx} + (1 - n)f(x)\cdot v = (1 - n)F(x)$, which is linear in v and x.

(6) Clairaut's equation: $y = xp + f(p)$, where $p = dy/dx$. The solution consists of the family of lines given by $y = Cx + f(C)$, where C is any constant, together with the curve obtained by eliminating p between the equations $y = xp + f(p)$ and $x + f'(p) = 0$, where $f'(p)$ is the derivative of $f(p)$.

Differential Equations of the Second Order

(7) $\dfrac{d^2y}{dx^2} = -n^2y$. Solution: $y = C_1 \sin(nx + C_2)$

or $y = C_3 \sin nx + C_4 \cos nx$

(8) $\dfrac{d^2y}{dx^2} = +n^2y.$ Solution: $y = C_1 \sinh (nx + C_2)$

or $y = C_3 e^{nx} + C_4 e^{-nx}$

(9) $\dfrac{d^2y}{dx^2} = f(y).$ Solution: $x = \displaystyle\int \frac{dy}{\sqrt{C_1 + 2P}} + C_2$, where $P = \int f(y)\, dy.$

(10) $\dfrac{d^2y}{dx^2} = f(x).$ Solution: $y = \displaystyle\int P dx + C_1x + C_2$, where $P = \displaystyle\int f(x)dx$

or $y = xP - \displaystyle\int xf(x)dx + C_1x + C_2$

(11) $\dfrac{d^2y}{dx^2} = f\left(\dfrac{dy}{dx}\right).$ Putting $\dfrac{dy}{dx} = z$, $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx}$, $x = \displaystyle\int \frac{dz}{f(z)} + C_1$ and $y = \displaystyle\int \frac{zdz}{f(z)} + C_2$; then eliminate z from these two equations.

(12) The equation for damped vibration: $\dfrac{d^2y}{dx^2} + 2b\dfrac{dy}{dx} + a^2y = 0.$

Case I. If $a^2 - b^2 > 0$, let $m = \sqrt{a^2 - b^2}$. Solution:

$y = C_1 e^{-bx} \sin (mx + C_2)$ or $y = e^{-bx}[C_3 \sin (mx) + C_4 \cos (mx)]$

Case II. If $a^2 - b^2 = 0$, solution is $y = e^{-bx}[C_1 + C_2x]$.

Case III. If $a^2 - b^2 < 0$, let $n = \sqrt{b^2 - a^2}$. Solution:

$y = C_1e^{-bx} \sinh (nx + C_2)$ or $y = C_3e^{-(b+n)x} + C_4e^{-(b-n)x}$

(13) $\dfrac{d^2y}{dx^2} + 2b\dfrac{dy}{dx} + a^2y = c.$ Solution: $y = \dfrac{c}{a^2} + y_1$, where y_1 = the solution of the corresponding equation with second member zero [see (12) above].

(14) $\dfrac{d^2y}{dx^2} + 2b\dfrac{dy}{dx} + a^2y = c \sin(kx).$ Solution:

$y = R \sin(kx - S) + y_1$, where $R = c/\sqrt{(a^2 - k^2)^2 + 4b^2k^2}$,

$\tan S = \dfrac{2bk}{a^2 - k^2}$, and y_1 = the solution of the corresponding equation with second member zero [see (12) above].

(15) $\dfrac{d^2y}{dx^2} + 2b\dfrac{dy}{dx} + a^2y = f(x).$ Solution: $y = y_0 + y_1$, where y_0 = any particular solution of the given equation, and y_1 = the general solution of the corresponding equation with second member zero [see (12) above].

If $b^2 > a^2$, $y_0 = \dfrac{1}{2\sqrt{b^2 - a^2}}\left\{ e^{m_1x} \displaystyle\int e^{-m_1x} f(x)dx - e^{m_2x} \int e^{-m_2x} f(x)dx \right\}$,

where $m_1 = -b + \sqrt{b^2 - a^2}$ and $m_2 = -b - \sqrt{b^2 - a^2}$.

If $b^2 < a^2$, let $m = \sqrt{a^2 - b^2}$; then $y_0 =$

$\dfrac{1}{m}e^{-bx}\left\{ \sin (mx) \displaystyle\int e^{bx} \cos (mx)\cdot f(x)dx - \cos (mx) \int e^{bx} \sin (mx)\cdot f(x)\, dx \right\}.$

If $b^2 = a^2$, $y_0 = e^{-bx}\left\{ x \displaystyle\int e^{bx} f(x)\, dx - \int x\cdot e^{bx} f(x)\, dx \right\}.$

GRAPHICAL REPRESENTATION OF FUNCTIONS

For graphical methods in statistics, etc., see W. C. Brinton's "Graphical Methods for Presenting Facts"

EQUATIONS INVOLVING TWO VARIABLES

The Curve $y = f(x)$. To represent graphically any function, y, of a single variable, x, lay off the values of x as **abscissae** along a uniformly graduated horizontal axis, whose positive direction (as usually chosen) runs to the right, and at each point on this x-axis erect a perpendicular (called an **ordinate**) whose length represents the value of y at that point. The unit of measurement for the y-scale, whose positive direction (as usually chosen) runs upward, need not be the same as the unit for the x-scale. Draw a smooth curve through the extremities of the ordinates; this is the **graph** of the given function in rectangular co-ordinates, or the **curve** of the function.

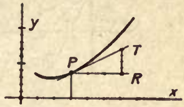


FIG. 1.

To measure graphically the rate of change of the function at any point P (Fig. 1), draw the tangent at P; then **rate of change** at $P = RT/PR$, where RT and PR are measured in units of the y-axis and x-axis, respectively. This ratio, which is positive if RT runs upward, negative if RT runs downward, is equal to the derivative of the function at the point P (see p. 157).

Graphs of Important Functions. Figs. 2–9 show the graphs (in rectangular co-ordinates) of the most important elementary functions, namely:

The **linear function**, $y = mx + b$ (Fig. 2).

The **power functions**, $y = x^n$ [n positive (parabolic type); n negative (hyperbolic type)] (Fig. 3).

The **exponential function**, $y = 10^x$ or $y = e^x$, and the **logarithmic function**, $y = \log_{10} x$ or $y = \log_e x$ (Fig. 4).

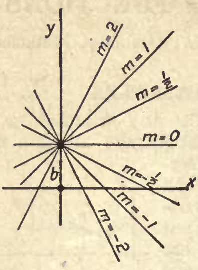
The **trigonometric functions** (Fig. 5), and the inverse trigonometric functions (Fig. 6).

The **hyperbolic functions** (Figs. 7 and 8) and the inverse hyperbolic functions (Fig. 9).

Various **special functions** (Figs. 10–12).

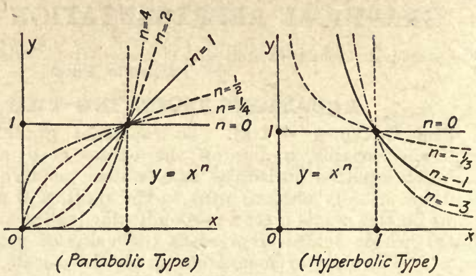
By a slight modification, each of these diagrams may be made to represent a somewhat more general function than that for which it is primarily intended. For, if x is replaced by $x - a$ in the equation, this merely requires re-numbering the x-axis so that each number is moved a units to the left; and similarly, if y is replaced by $y - b$ in the equation, this merely requires re-numbering the y-axis so that each number is moved b units downward. (Such a change is called a translation of the curve to the right, or upward.) Further, if x is replaced by x/c [or y by y/c] in the equation, it is merely necessary to multiply each of the numbers written along the x-axis [or y-axis] by c, in order to adapt the graph to the new equation. (Such a change is called a "stretching" of the curve along one of the axes.)

Empirical Curves. Any set of values of two variables x and y can be represented by plotting the points (x,y) on rectangular co-ordinate paper, and drawing a smooth curve through these points. The points which correspond to actual data should be clearly indicated by small circles or crosses, intermediate points being spoken of as interpolated points. While this process of graphically interpolating a continuous series of points *between* given values is usually fairly safe, the process of extrapolation—that is, extending the curve *beyond* the range of the given values, is dangerous.



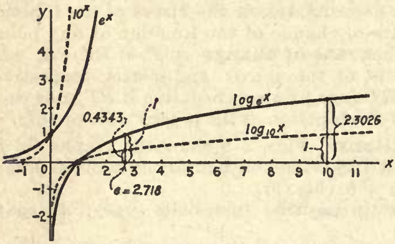
Linear function, $y = mx + b$.

FIG. 2.



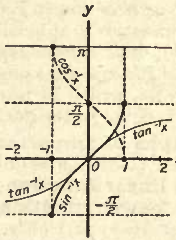
Power function, $y = x^n$.

FIG. 3.



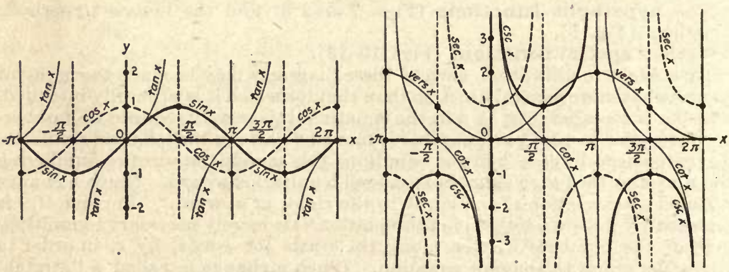
Exponential function (10^x or e^x).
Logarithmic function ($\log_{10} x$ or $\log_e x$).

FIG. 4.



Inverse trigonometric functions.

FIG. 6.

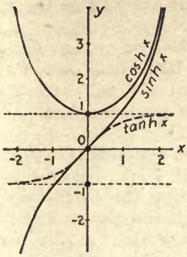


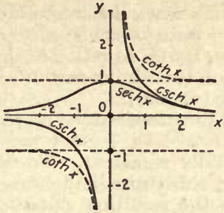
Trigonometric functions.

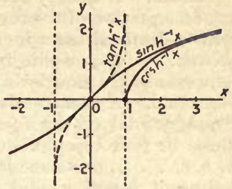
FIG. 5.

To Find a Mathematical Equation to Fit a Given Empirical Curve. This problem is one which in general requires much patience and ingenuity. Only the simplest cases can be mentioned here.

CASE 1. If the given empirical curve is a straight line, then the law connecting the given values of x and y is $y = mx + b$, where m = the slope of the line, and b = the value of y at the point where the line crosses the y-axis. If

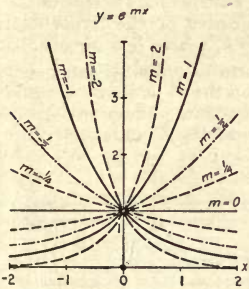


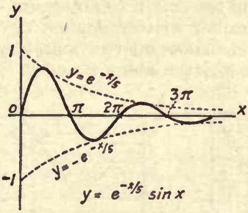




Hyperbolic functions and inverse hyperbolic functions.

FIG. 7. FIG. 8. FIG. 9.





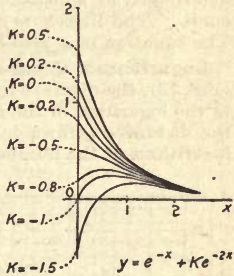


FIG. 10. FIG. 11. FIG. 12.

the points lie only approximately on a straight line, the best position for this line can usually be found by stretching a black thread among the points; or, assume a law of the form $y = mx + b$, and, by substituting in this formula n pairs of values of x and y, obtain n equations connecting the coefficients m and b; various pairs of these equations may then be solved for m and b, and the average of the results taken. Or, if great accuracy is required, all n of the equations may be solved for m and b by the method of least squares (p. 121).

If any law of the form $f(x,y) = m \cdot F(x,y) + b$ is suspected, where $f(x,y)$ and $F(x,y)$ are any expressions involving either x or y or both x and y, such a law may be tested by plotting $F(x,y)$ instead of x, and $f(x,y)$ instead of y, on rectangular cross-section paper, and seeing whether or not the points lie on a straight line. If they do, the form of the law is verified, and the values of m and b can be read from the figure as before. For example, if $y^2 = mxy + b$, a straight line will be obtained by plotting y^2 against xy. Again, if $xy = bx + my$, a straight line will be obtained by plotting y against y/x, since the equation may be written $y = b + m\,(y/x)$.

CASE 2. If a law of the form $y = cx^n$ is suspected, plot the points (x,y) on logarithmic paper (see below).

CASE 3. If a law of the form $y = c \cdot 10^{mx}$ [or $y = c \cdot e^{mx}$] is suspected, plot the points (x,y) on semi-logarithmic paper (see below).

CASE 4. If the given curve resembles the logarithmic curve, $y = \log x$, interchange x and y and proceed as in Case 3.

CASE 5. If the given curve is a wavy line, resembling a sine or cosine curve, try an equation of the form $y = a \sin bx$ or $y = a \cos bx$. If the heights of the waves diminish as x increases, try an equation of the form $y = ae^{-nx} \sin bx$. [NOTE. Any periodic function (satisfying certain simple conditions) can be expressed by a Fourier's series (p. 162)].

CASE 6. A great variety of functions can be represented approximately by a polynomial of the form $y = a + bx + cx^2 + dx^3 + ex^4 + . . .$, the first three or four terms being usually sufficient. To determine the coefficients $a, b, c, . . .$, most accurately, substitute in the formula all the given pairs of values of x and y, and solve the resulting equations for $a,b,c, . . .$ by the method of least squares (p. 121).

CASE 7. Many simple curves can be represented approximately by an equation of the hyperbolic form, $xy = c + bx + ay$, where a, b, and c are determined by substituting the co-ordinates of three conspicuous points of the curve. The lines $x = a$ and $y = b$ are the asymptotes of the hyperbola. The equation may also be written $(x - a)(y - b) = k$, where $k = ab + c$.

Logarithmic Cross-section Paper. In this form of cross-section paper (Fig. 13), the distance from the origin to any point on the x- or y-axis is equal to the logarithm of the number written against that point. Thus, in Fig. 13 the distances (shown for clearness on two auxiliary scales X and Y) are the logarithms of the numbers written along x and y.

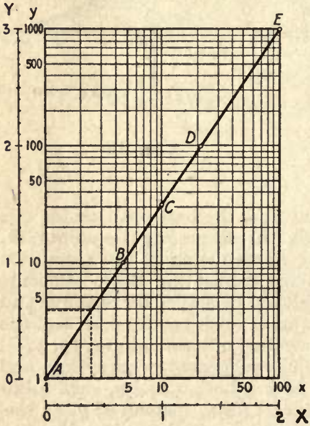


FIG. 13.

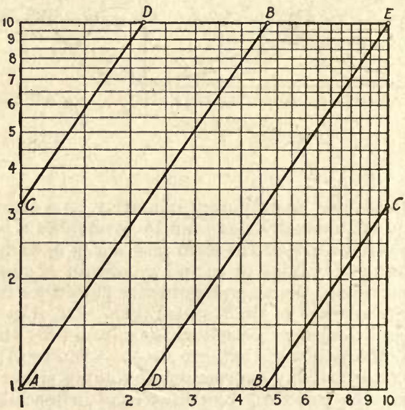


FIG. 14.

Accurately made logarithmic paper can be obtained from the principal dealers in draftmen's supplies. Logarithmic paper can be easily constructed, in case of need, by copying the logarithmic scale from any ordinary slide rule. The actual figures along the x- and y-axes are usually left for the user to insert; in so doing, notice that the numbers . . ., 0.01, 0.1, 1, 10, 100, . . ., or such of them as may be needed to cover any given range of values, must be placed at the points of division which separate the main squares. It is often convenient, however, to omit the decimal point, num-

bering each square independently from 1 to 10. The length of the side of one square is called the *unit* or *base* of the logarithmic paper; the larger the unit, the finer the possible subdivisions of the scale.

To plot a point (x,y) on logarithmic paper, for example, the point (3,5), means to find the point of intersection of the vertical line marked $x = 3$ and the horizontal line marked $y = 5$. In interpolating between two lines, account should be taken of the fact that the divisions are not of uniform length.

Any equation of the form $y = cx^n$ when plotted on logarithmic paper will be represented by a straight line whose slope is n. For, if $y_1 = cx_1^n$ and $y_2 = cx_2^n$, then $y_1/y_2 = (x_1/x_2)^n$, or $(\log y_1 - \log y_2)/(\log x_1 - \log x_2) = n$. The slope must be measured by aid of an auxiliary *uniform* scale.

EXAMPLE. Let $y = x^{3/2}$. When $x = 1$, $y = 1$; plot this point A on the logarithmic paper, and draw the straight line AE with a slope equal to $3/2$ (Fig. 13). By the aid of this line, the value of y for any value of x between 1 and 100 can be read off directly; for example, if $x = 2.50$, $y = 3.95$, as shown by dotted lines, so that $(2.50)^{3/2} = 3.95$. To find the value of y for any value of x outside this range, note that moving the decimal point 2 places in x is equivalent to moving it 3 places in y. The line shown in Fig. 13 is thus equivalent to a complete table of three-halves powers.

It will be noticed that this line crosses four squares of the logarithmic paper. By superposing these four squares the whole diagram may be condensed into a single square (Fig. 14), in which, however, the scales for x and y now give only the sequence of digits in the answer, the position of the decimal point having to be determined by inspection.

To determine whether a given set of values, x and y, satisfies a law of the form $y = cx^n$, plot the values on logarithmic paper, and see whether they lie on a straight line; if they do, then the given values satisfy a law of this form; moreover, the slope of the line gives the value of n, and the value of y when $x = 1$ gives the value of c.

If the plotted points fail to lie exactly in line, but form a curve slightly concave upward, try subtracting some constant b from all the y's, that is, move each point downward a distance equal to b units of the y-scale at that point. If it proves possible to choose b so that the resulting points lie in line, then the original values obey a law of the form $y - b = cx^n$, where n is again the slope of the line, and c is the value of $y - b$ when $x = 1$. (Conversely, if the curve is concave *downward*, try *adding* b to all the y's; that is, move each point *upward;* if the new points lie in line, the original values obey a law of the form $y + b = cx^n$.) Another method of "straightening" the curve consists of adding some constant, $\pm a$, to all the values of x, which has the effect of shifting all the points to the right or left (by varying amounts); if this method succeeds, the original values obey a law of the form $y = c(x + a)^n$.

Semi-logarithmic Cross-section Paper*. This form of paper (Fig. 15) has a logarithmic scale along y and a uniform scale along x. The **"scale value,"** k, of the paper is the number which stands, on the x-axis, at a distance from the origin equal to the width of one of the main horizontal strips. Thus, in Fig. 15, each number shown along the auxiliary scale Y is the logarithm of the corresponding number along y, and each number shown along the auxiliary scale X is $1/k$th of the corresponding number along x (here $k = 5$). The number k, which may be chosen at pleasure, should be taken equal to some simple integer, as 1, 2, or 5, or some integral power of 10.

In preparing the paper for use it is important to notice that the numbers . . ., 0.01, 0.1, 1, 10, 100, . . . (or such of them as may be needed in any given case) must be placed along the y-axis at the points which mark the main lines of division between the horizontal strips; while the numbers . . ., $-2k$, $-k$, 0, $+k$, $+2k$, . . . (or such of them as may be needed) must be placed along the x-axis at uniform intervals, each interval (from 0 to k, from k to $2k$, etc.) being equal to the width of one of the main horizontal strips. The width of one of these strips is called the *unit* or *base* of the semi-

* Made by the Educational Exhibition Co., 26 Custom House St., Providence, R. I.

logarithmic paper; the larger the unit, the finer the possible subdivisions of the scale.

To plot a point (x,y), as $x = 3$, $y = 5$, on semi-logarithmic paper means to find the point of intersection of the vertical line marked $x = 3$ with the horizontal line marked $y = 5$.

Any equation of the form $y = c \cdot 10^{mx}$ [or $y = c \cdot e^{mx}$] when plotted on semi-logarithmic paper with scale value k, will be represented by a straight line whose slope is km [or $0.4343\ km$.]. By a suitable choice of the scale value k, any given range of values of x can be brought within the size of the paper. Note that $e = 10^{0.4343}$.

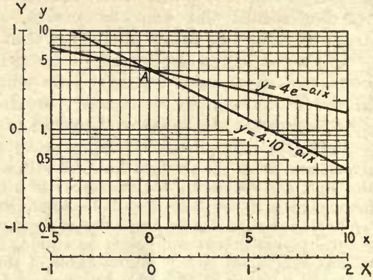


FIG. 15.

EXAMPLE. Given $y = 4 \cdot 10^{-0.1x}$ [or $y = 4 \cdot e^{-0.1x}$]. In Fig. 15, when $x = 0$, $y = 4$. By plotting this point (A) on the semi-logarithmic paper, with scale value 5, and drawing through it a straight line with slope equal to -0.5 [or -0.217] a graphical representation is obtained from which, for any value of x, the corresponding value of y can be read off. If it is desired to condense the figure, several horizontal strips may be superposed on a single strip; this of course renders the decimal point in the y-scale undetermined (unless a separate y-scale is provided for each section of the graph).

In order to determine whether a given set of values of x and y satisfy a law of the form y = c·10^{mx} [or y = c·e^{mx}], plot the values of x and y on semi-logarithmic paper, with a suitable scale value k, and see whether they lie on a straight line; if they do so, the law is satisfied, and the values of m and c may be found as follows: m = the slope of the line divided by k [or the slope of the line divided by $0.4343k$], and c = the value of y when $x = 0$.

If the plotted points fail to lie exactly in line, but form a curve slightly concave upward, try subtracting some constant b from all the y's, and plot the values thus modified; if b can be so chosen that the revised points lie in line, then the original values obey a law of the form $y - b = c \cdot 10^{mx}$ [or $y - b = c \cdot e^{mx}$], where m and c are to be found as before. If the curve is concave downward, add b, instead of subtracting; and replace $y - b$ by $y + b$ in the law.

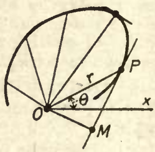


FIG. 16.

Curves in Polar Co-ordinates. Any function, r, of a single variable, θ, can be represented by a curve in polar co-ordinates (p. 137). Lay off the given values of θ as angles, the initial line Ox running toward the right, and the counterclockwise direction about the origin being taken as positive. Along the terminal side of each angle θ, lay off the corresponding value of r, forward if r is positive, backward if r is negative; and pass a smooth curve through the points thus determined.

The rate of change of r with respect to θ at a given point P is represented graphically as follows (Fig. 16): On the tangent at P drop a perpendicular OM from the origin; then $r(MP/OM)$ represents the rate of change, $dr/d\theta$, provided θ is measured in radians. Specially ruled polar co-ordinate paper is supplied by dealers in drafting supplies.

EQUATIONS INVOLVING THREE VARIABLES

The Surface z = f(x, y). Any function, z, of two variables, x and y, may be represented by a surface, as follows: Plot the given pairs of values of x and y as points in a horizontal x, y plane, called the base plane; at each of these points erect an ordinate, parallel to a vertical axis z, and representing

by its length the value of z at that point. Then conceive a smooth surface passed through the extremities of these ordinates: this surface is said to represent the function. In practice, the ordinates may be made by implanting stiff vertical rods in a horizontal board of soft wood which serves as the base plane; the surface may then be constructed by filling in the spaces with plaster of Paris. Or, more simply, pieces of cardboard may be cut out to represent parallel plane sections of the surface, and then stood on edge in slots cut in the board to receive them. The units employed along x, y, and z need not be equal to each other.

Contour-line Charts. All the points of a surface $z = f(x, y)$ which are at any given height above the base plane form a curve on the surface, called a contour line of the surface. If each of these contour lines be projected on the base plane, and each labeled with the value of z to which it corresponds, a complete representation of the function $z = f(x, y)$ is obtained, all in one plane. A topographical map, with contour lines showing elevations above the sea, and a weather map, with contour lines showing barometric pressure, are familiar examples. If there are several values of z corresponding to any given point (x, y), there will be several contour lines whose projections pass through that point.

Contour-line Charts for Simultaneous Equations [of the form $z = f(x,y)$, $w = F(x,y)$]. In Fig. 17, plot the function $z = f(x,y)$ by contour lines on an x,y plane, and plot the function $w = F(x,y)$ by contour lines on the same x,y plane. Then every point on the diagram (either directly or by interpolation) is the intersection of four curves—an x-curve, a y-curve, a z-curve, and a w-curve. Here, by "curve" is meant any line, straight or curved. By the aid of such a diagram, when the values of any two of these four variables are given, the values of the other two can be found. The method of use consists simply in entering the diagram along the two given curves (or lines), tracing them to their point of intersection, and then coming out again along the two curves (or lines) whose values are required. The best manner of numbering the curves is indicated in the figure.

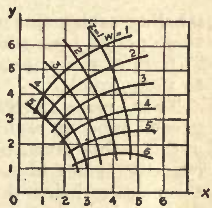


FIG. 17.

Alignment Charts for Three Variables, t, u, v. Any relation between three variables, t, u, v, which can be thrown into one of the forms listed in later paragraphs, can be represented graphically by a very convenient form of diagram called an alignment chart. In the simplest form of an alignment chart for three variables there are three scales (straight or curved), along which the values of the three variables, t, u, v, are marked in such a way that any three values of t, u, v which satisfy the given equation are represented by three points which lie in line. Hence, if the values of any two of the variables are given, the corresponding value of the third can be found by simply drawing a straight line through the two given points and reading the value of the point where it crosses the third scale.

The most important methods of constructing alignment charts for three variables are described below. Where several methods are applicable in a given case, the best one must be determined largely by trial. For further information see M. d'Ocagne, "Traité de Nomographie" (Gauthier-Villars, Paris); Carl Runge, "Graphical Methods" (Columbia University Press); J. B. Peddle, "Construction of Graphical Charts" (McGraw-Hill); see also page 185.

Notation. In each of the equations which follow, U stands for any function of u alone, V for any function of v alone, and $F_1(t)$, $F_2(t)$ for any functions of t alone. Any of these functions may reduce to a constant. The axes of x, y, and y' which are mentioned are of merely temporary use in constructing the diagram, and the letters x, y, y' should not be written on the chart. It is not necessary that the axes be at right angles, provided the x of a point is always measured parallel to the x-axis, and its y parallel to the y-axis.

Method 1. Given, an equation which can be thrown into the form

$$U \cdot F_1(t) + V \cdot F_2(t) = 1,$$

where, for the given range of values of u and v, the largest variations in U and V are less than a certain number m.

Draw a pair of (temporary) x,y axes (Fig. 18), and through the point $x = 1$ draw a third axis, which may be called the axis of y', parallel to the axis of y. In ordinary cases, the unit of measurement along x should be nearly equal to the full width of the paper. Now choose a unit for y and y' such that m times this unit will about equal the height of the paper, and plot, in the usual way, the points (x,y) given by

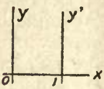


FIG. 18.

$$x = \frac{F_2(t)}{F_1(t) + F_2(t)}, \quad y = \frac{1}{F_1(t) + F_2(t)},$$

labeling each point with the value of t to which it corresponds. Connect these points by a smooth curve, which gives the t-scale of the diagram. [If $F_1(t)/F_2(t)$ = a constant, the t-scale will prove to be a straight line parallel to the y-axis.]

Then, using the same units as above, plot along y the points given by $y = U$, labeling each point with the corresponding value of u; and plot along y' the points given by $y' = V$, labeling each of these points with the corresponding value of v. This gives the u- and v-scales of the diagram. The three scales being thus constructed, the x-axis may now be erased, and the diagram is ready for use. Any three points t, u, v which lie in line correspond to three values of t, u, v, which satisfy the given equation. The numbering on each scale should be shown at sufficiently frequent intervals to permit of easy interpolation.

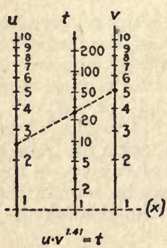


FIG. 19.

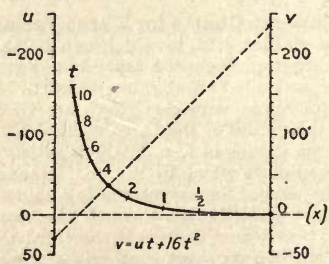


FIG. 20.

EXAMPLE 1 (Fig. 19). Let $uv^{1.41} = t$. By taking the logarithm of both sides, and dividing through by log t, reduce the equation to the form $(\log u)(1/\log t) + (\log v) \times (1.41/\log t) = 1$. Here $U = \log u$, $V = \log v$, $F_1(t) = 1/\log t$, $F_2(t) = 1.41/\log t$, and $x = 1.41/2.41 = 0.585$, $y = (1/2.41)\log t$.

EXAMPLE 2 (Fig. 20). Let $v = ut + 16t^2$, which reduces to the form $(-u/16)(1/t) + (v/16)(1/t^2) = 1$. Here $U = -u/16$, $V = v/16$, $F_1(t) = 1/t$, $F_2(t) = 1/t^2$ and $x = 1/(1+t)$, $y = t^2/(1+t)$.

NOTE. If $m = \infty$, values of u and v which give large values of U and V cannot be shown within the limits of the paper. In such cases, the chart may be supplemented by a second chart, made according to Method 2, below.

Method 2. Given, an equation which can be thrown into the form

$$\frac{F_1(t)}{U} + \frac{F_2(t)}{V} = 1,$$

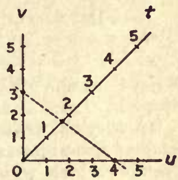
where, for the given range of values of u and v, the largest variation in U is less than a certain number m. and the largest variation in V is less than a certain number n.

Draw a pair of temporary x,y axes, and having chosen a unit for the x-axis equal to about $(1/m)$th of the width of the paper, and a unit for the y-axis equal to about $(1/n)$th of the height, plot the points (x,y) given by

$$x = F_1(t),\ y = F_2(t),$$

labeling each point of this curve with the value of t to which it corresponds. Connect these points by a smooth curve, which gives the t-scale of the diagram. [If $F_1(t)/F_2(t)$ = a constant, the t-scale will be a straight line through the origin.]

Then, using the same units as above, plot along x the values of U, labeling each point with the corresponding value of u; and plot along y the values of V, labeling each point with the corresponding value of v. This gives the u- and v-scales of the diagram. On the chart as thus completed, any three points t, u, v which lie in line correspond to three values of t, u, v which satisfy the given equation.



$t = uv/(u+v)$

FIG. 21.

EXAMPLE (Fig. 21). Let $t = (uv)/(u + v)$, which may be written in the form $t/u + t/v = 1$. Here $U = u$, $V = v$, $F_1(t) = t$, $F_2(t) = t$.

NOTE. If $m = \infty$ and $n = \infty$, values of u and v which give large values of U and V cannot be shown within the limits of the paper. In such cases the chart may be supplemented by a second chart, made according to Method 1, above.

Method 3. Given, an equation which can conveniently be thrown into the form

$$F_2(t) = V \cdot F_1(t) + U,$$

where, for the given range of values of t, the largest variation in $F_1(t)$ is less than a certain number m, and the largest variation in $F_2(t)$ is less than a certain number n.

Draw a pair of temporary x,y axes, and, having chosen a unit for x equal to about $(1/m)$th of the width of the paper and a unit for y equal to about $(1/n)$th of the height, plot the points (x,y) given by

$$x = F_1(t),\ y = F_2(t),$$

labeling each point of the curve with the value of t to which it corresponds. Connect these points by a smooth curve, which forms the t-scale. Next, using the same unit for y as above, plot along the y-axis the values of U, labeling each point with the corresponding value of u. This gives the u-scale. Finally, with the origin as center, and any convenient radius, draw a circle cutting the x-axis in A. Along this circular arc, starting from A in the counterclockwise direction, lay off the angles whose slopes are equal to V, labeling each point of the arc with the value of v to which it corresponds.

This gives the v-scale, which in this case, however, plays a peculiar rôle, since, in using this form of chart, two straight lines are required instead of one. Thus:

In order to determine whether three values, t, u, v, satisfy the given equation, lay one straight line through the points t and u, and another straight line through the point v and the origin; if these lines are parallel, the three values of t, u, v satisfy the equation. It will be noticed that the function of the v-scale here is to measure, in a certain sense, the slope of the line joining t and u. A chart of this type may be called "an alignment chart with a sliding scale for one of the variables."

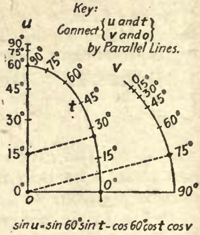


FIG. 22.

EXAMPLE (Fig. 22). Let $\sin u = \sin 60° \sin t - \cos 60° \cos t \cos v$, which may be put in the form

$$(\sin 60° \sin t) = \cos v\,(\cos 60° \cos t) + \sin u.$$

Here $F_1(t) = \cos 60° \cos t$, $F_2(t) = \sin 60° \sin t$, $U = \sin u$, $V = \cos v$.

Method 4. Given, an equation which can be reduced to the form

$$U \cdot F(t) + V = 0,$$

where, for the given range of values of u and v, the largest variations in U and V are less than a certain number m.

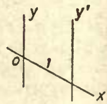


FIG. 23.

In Fig. 23, draw temporary axes x, y, and y', and choose the units as in Method 1. To construct the t-scale, which will now coincide with the x-axis, plot along x the points for which

$$x = \frac{1}{1 + F(t)},$$

labeling each point with the value of t to which it corresponds. The u-scale, along the axis of y, and v-scale, along the axis of y', are constructed exactly as in Method 1, and the finished chart is used in the same way.

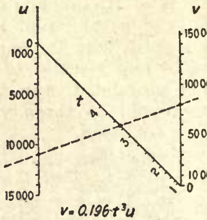


FIG. 24.

EXAMPLE (Fig. 24). Let $v = 0.196\ t^3 u$, where u is to range from 0 to 15,000 and v from 0 to 150,000. The equation may be written in the form $(-10\,u)(0.0196t^3) + v = 0$. Here $U = -10\,u$, $V = v$, $F(t) = 0.0196t^3$.

NOTE. If $m = \infty$, values of u and v which give large values of U and V cannot be shown within the limits of the paper.

EQUATIONS INVOLVING FOUR VARIABLES

[For simultaneous equations of the form $z = f(x,y)$, $w = F(x,y)$, see p. 179.]

Alignment Charts for Four Variables. The extension of the methods of the alignment chart to the case of four variables, say r, s, u, v, consists essentially in replacing the t-scale of the earlier diagram by a network of two scales, one for r and one for s. The point where a curve $r = r_1$ and a curve $s = s_1$ intersect may be spoken of as the point (r_1,s_1). In the following equations, U denotes as before any function of u alone, V any function of v alone; while $F_1(r,s)$ and $F_2(r,s)$ represent any functions of r and s.

Method 1a. Given, an equation of the form

$$U \cdot F_1(r,s) + V \cdot F_2(r,s) = 1.$$

Draw axes x, y, and y' as in Method 1, and plot the network of curves given by the equations

$$x = \frac{F_2(r,s)}{F_1(r,s) + F_2(r,s)}, \qquad y = \frac{1}{F_1(r,s) + F_2(r,s)}.$$

[To do this (Fig. 25), find the point (x,y) that corresponds to each given pair of values of r and s, by direct substitution in the equations for x and y. Connect all the points for which $r = 1$ by a curve, and label it $r = 1$; connect all the points for which $r = 2$ by another curve, and label it $r = 2$; etc. This gives the family of r-curves. Similarly, through all the points for which $s = 1$ draw a curve labeled $s = 1$; through all the points for which $s = 2$ draw a curve labeled $s = 2$; etc. This gives the family of s-curves, intersecting the family of r-curves. Note, however, that if it is possible to eliminate s (or r) from the equations that give x and y, the resulting equation in x, y, and r (or x, y, and s) can often be plotted directly for each given value of r (or of s).]

Next, construct the u- and v-scales along the axes of y and y' as in Method 1. [The letters x, y, and y', and the units used in plotting along these axes, should be omitted from the finished diagram, as should also the axis of x.]

In the chart, as thus completed, any three points, (r,s), u, and v which lie in a straight line, correspond to values of r, s, u, v which satisfy the given equation. Hence, when any three of these four values are given, the fourth can be found from the chart.

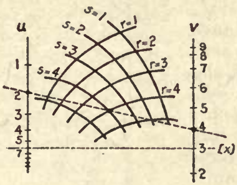


FIG. 25.

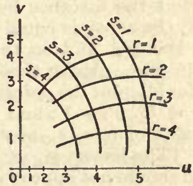


FIG. 26.

Method 2a. Given, an equation of the form

$$\frac{F_1(r,s)}{U} + \frac{F_2(r,s)}{V} = 1.$$

Draw axes of x and y as in Method 2, and plot the network of curves given by

$$x = F_1(r,s), \quad y = F_2(r,s).$$

To do this, follow the plan outlined for a similar case under Method 1a, labeling each curve of the r-family (Fig. 26) with the corresponding value of r, and each curve of the s-family with the corresponding value of s. Next, construct the u- and v-scales along the x- and y-axes, precisely as in Method 2. Then any three points, (r,s), u, and v, which lie in a straight line correspond to values of r, s, u, v which satisfy the given equation.

Method 3a. Given, an equation of the form

$$F_2(r,s) = V \cdot F_1(r,s) + U.$$

Draw axes of x and y, as in Method 3, and plot the network of curves given by $x = F_1(r,s)$, $y = F_2(r,s)$, following the plan outlined for a similar case under Method 1a, and labeling each curve of the r-family (or s-family) with the value of r (or s) to which it corresponds. Next, construct the u-scale

along the y-axis, and the v-scale along a circular arc, precisely as in Method 3. Then any three points, (r,s) u, and v, which are so related that the line through (r,s) and u is parallel to the line joining v with the origin, will correspond to values of r, s, u, v which satisfy the given equation.

EXAMPLE for Method 3a (Fig. 27). Let $\cot v = \cot r \cos s + \csc r \sin s \cot u$, which may be written $(\cos r \cot s) = \cot v (\sin r \csc s) - \cot u$. Here $U = -\cot u$, $V = \cot v$, $F_1(r,s) = \sin r \csc s$, $F_2(r,s) = \cos r \cot s$, whence $\dfrac{x^2}{\csc^2 s} + \dfrac{y^2}{\cot^2 s} = 1$, $\dfrac{x^2}{\sin^2 r} - \dfrac{y^2}{\cos^2 r} = 1$, so that the s-curves are ellipses and the r-curves hyperbolas.

Parallel Charts, or Proportional Charts, for Four Variables. In the following methods of representation there are four scales, one for each of the four variables, and the method of using the diagram consists in connecting two pairs of points by parallel lines.

Method A. Given, an equation of the form

$$R - S = U - V$$

where R, S, U, V are any functions of the variables r, s, u, v, respectively. [It will be noted that any proportion $R/S = U/V$ can at once be thrown into this form by taking the logarithm of both sides.]

In Fig. 28, draw four vertical axes, y_1, y_2, y'_1, y'_2, such that the distance between y_1 and y'_1 (which may be zero) is equal to the distance beween y_2 and y'_2, and so that the four zero points lie in line. Along these axes, using the same unit for all, plot the points given by $y_1 = R$, $y'_1 = S$, $y_2 = U$, $y'_2 = V$, and label each point with the value of r, s, u, or v to which it corresponds. (The letters y_1, y_2, y'_1, y'_2 are temporary, and should not appear on the diagram.) Then if the line joining two points r and u is parallel to the line joining two points s and v, the four values of r, s, u, v will satisfy the given equation. In this and the following methods, a parallel ruler, or a pair of draftman's triangles, will be useful in reading the chart. A "key" stating which points are to be joined with which, should be clearly given on the diagram.

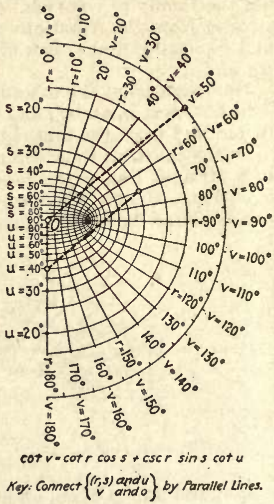


FIG. 27.

EXAMPLE (Fig. 28). Let $32.2\, vr = us^2$, or $\log r - 2 \log s = \log u - \log (32.2\, v)$. Here $R = \log r$, $S = 2 \log s$, $U = \log u$, $V = \log (32.2\, v)$.

Method B. Given, an equation of the form

$$\frac{R}{S} = \frac{U}{V}$$

In Fig. 29, draw a pair of axes, x,y, and parallel to them (or coinciding with them) a second pair of axes, x_1,y_1. Using any convenient horizontal unit, plot along x and x_1 the points given by $x = R$, $x_1 = U$, and using any convenient vertical unit, plot along y and y_1 the points given by $y = S$, $y_1 = V$. Label each point with the value of r, s, u, v, to which it corresponds. (The letters x, y, x_1, y_1 should not appear on the diagram.) Then if the line joining two points r and s is parallel to the line joining two points u and v, the four values r, s, u, v will satisfy the given equation.

Method C. Given, an equation of the form

$$R - S = \frac{V}{U}.$$

In Fig. 30, take a pair of axes, x,y, and through the point $x = 1$ draw a third axis, y', parallel to y. Also, take a second pair of axes, x_2,y_2, parallel to (or coinciding with) the axes of x and y. Having chosen a suitable unit for x and x_2, and a suitable unit for y, y', and y_2, lay off the values of R and

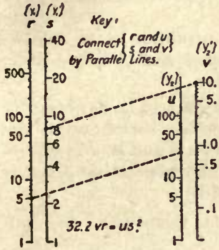


FIG. 28.

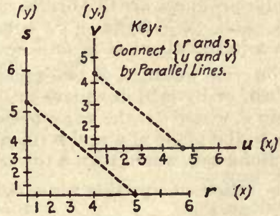


FIG. 29.

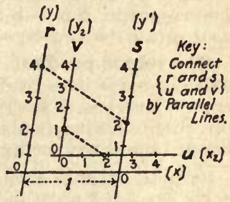


FIG. 30.

S along y and y', respectively, labeling each point with the value of r or s to which it corresponds; and lay off the values of U and V along x_2 and y_2, labeling each point with the value of u or v to which it corresponds. Then if the line joining two points r and s is parallel to the line joining two points u and v, the four values r, s, u, v will satisfy the given equation. This form of chart is sometimes called a "Z-chart."

For further examples, see R. C. Strachan, "Nomographic Solutions for Formulas of Various Types," Trans. Am. Soc. Civil Engineers, vol. 78, 1915.

VECTOR ANALYSIS

Many problems involving directed magnitudes can be advantageously treated by the methods of vector analysis. The following is a brief summary of the principal definitions and formulæ.

A set of arrows, each arrow having a given *length* and pointing in a given *direction*, is called a set of **vectors**, provided they combine by addition according to the parallelogram law (see below). Notation: $\mathbf{a}$ or $\bar{a}$ for a vector; a or $|\mathbf{a}|$ for its length. Two "free" vectors are equal if they have the same length and point in the same direction; two "sliding" vectors are equal if they have the same length and direction, and also lie in the same line.

A **scalar** is any real number, positive, negative, or zero.

Addition of vectors.—If an arrow $\mathbf{a}$ is immediately followed, tip to tail, by a second arrow $\mathbf{b}$, then the arrow which runs from the beginning of $\mathbf{a}$ to the end of $\mathbf{b}$ is called the **sum** of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} + \mathbf{b}$. Conversely, if $\mathbf{a} + \mathbf{x} = \mathbf{b}$, then $\mathbf{x} = \mathbf{b} - \mathbf{a}$. The laws of operation for $+$ and $-$ are the same as in ordinary algebra (pp. 112, 124). If m is a scalar, then $m\mathbf{a}$ means a vector having the same direction as $\mathbf{a}$, and m times its length.

Multiplication of vectors is of two kinds, as follows:

The **scalar product,** or dot product, of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\cdot\mathbf{b}$—or sometmes by $S\mathbf{ab}$, or by $(\mathbf{ab})$ in round parentheses—is defined as the scalar quantity $ab \cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

EXAMPLE. If $\mathbf{F}$ is a force whose point of application moves along a vector distance $\mathbf{x}$, then $\mathbf{F}\cdot\mathbf{x}$ = *work* done by $\mathbf{F}$ during this displacement.

Peculiarities of scalar products: (1) Since $\mathbf{a}\cdot\mathbf{b}$ is not a vector, expressions like $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ will not occur; (2) from $\mathbf{a}\cdot\mathbf{x} = \mathbf{a}\cdot\mathbf{y}$ we cannot infer that $\mathbf{x} = \mathbf{y}$, hence, quotients will not occur; (3) from $\mathbf{a}\cdot\mathbf{b} = 0$, it follows that $\mathbf{a}$ is **perpendicular** to $\mathbf{b}$ (unless $\mathbf{a}$ or $\mathbf{b}$ is zero).

On the other hand, scalar products are like ordinary products in the following respects: $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$, and $(\mathbf{a} + \mathbf{b})\cdot(\mathbf{c} + \mathbf{d}) = \mathbf{a}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d}$; also, $m(\mathbf{a}\cdot\mathbf{b}) = (m\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(m\mathbf{b})$, where m is any scalar.

The **vector product,** or cross product, of two vectors $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a}\times\mathbf{b}$—or sometimes by $V\mathbf{ab}$, or by $[\mathbf{ab}]$ in square brackets—is defined as the *vector* whose length is $ab \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and whose direction is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ (in such a sense that a right-handed screw advancing along $\mathbf{a}\times\mathbf{b}$ would turn $\mathbf{a}$ toward $\mathbf{b}$).

EXAMPLE. If $\mathbf{F}$ is a force acting on a particle whose radius vector is $\mathbf{r}$, then $\mathbf{r}\times\mathbf{F}$ = the *torque* of $\mathbf{F}$ about the origin.

Peculiarities of vector products: (1) $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, so that the order of the factors is always important; (2) $\mathbf{a}\times\mathbf{a} = 0$; (3) it is not true that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$; (4) from $\mathbf{a}\times\mathbf{x} = \mathbf{a}\times\mathbf{y}$ it does not follow that $\mathbf{x} = \mathbf{y}$; hence, quotients will not occur; (5) from $\mathbf{a}\times\mathbf{b} = 0$, it follows that $\mathbf{a}$ and $\mathbf{b}$ are **parallel** (unless $\mathbf{a}$ or $\mathbf{b}$ is zero).

On the other hand, as in ordinary algebra

$$(\mathbf{a} + \mathbf{b})\times(\mathbf{c} + \mathbf{d}) = \mathbf{a}\times\mathbf{c} + \mathbf{a}\times\mathbf{d} + \mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{d},$$

provided the order of factors in each product is preserved; also, $m(\mathbf{a}\times\mathbf{b}) = (m\mathbf{a})\times\mathbf{b} = \mathbf{a}\times(m\mathbf{b})$, where m is any scalar. Further laws are:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}); \text{ and } \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

Vector Differentiation. If $\mathbf{r} = \mathbf{f}(t)$ gives a vector $\mathbf{r}$ as a function of a scalar t, then $d\mathbf{r}/dt = \lim\{[\mathbf{f}(t + \Delta t) - \mathbf{f}(t)]/\Delta t\}$ as Δt approaches zero.

$$d(\mathbf{a} + \mathbf{b}) = d\mathbf{a} + d\mathbf{b}, \quad d(m\mathbf{a}) = m(d\mathbf{a}) + (dm)\mathbf{a},$$

$$d(\mathbf{a}\cdot\mathbf{b}) = (d\mathbf{a})\cdot\mathbf{b} + \mathbf{a}\cdot(d\mathbf{b}), \quad d(\mathbf{a}\times\mathbf{b}) = (d\mathbf{a})\times\mathbf{b} + \mathbf{a}\times(d\mathbf{b}).$$

EXAMPLE. If $\mathbf{r} = \mathbf{f}(t)$ gives the position-vector of a moving particle as a function of the time t, then $d\mathbf{r}/dt$ = its vector velocity, $\mathbf{v}$, and $d\mathbf{v}/dt$ = its vector acceleration, $\mathbf{a}$. If $\mathbf{m}$ and $\mathbf{n}$ are unit vectors in the direction of the tangent and normal to the path at the time t, then $\mathbf{v} = v\mathbf{m}$, where $v = ds/dt$ = the (scalar) path-velocity, and $d\mathbf{m} = [(ds/R)]\mathbf{n}$, where R = the (scalar) radius of curvature of the path. Then

$$\mathbf{a} = \frac{d(v\mathbf{m})}{dt} = \frac{dv}{dt}\mathbf{m} + v\frac{d\mathbf{m}}{dt} = \frac{dv}{dt}\mathbf{m} + \frac{v^2}{R}\mathbf{n}.$$

Here dv/dt and v^2/R are the familiar expressions for the components of acceleration along the tangent and normal.

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